CATEGORICAL EMBEDDINGS
AND
LINEARIZATIONS

By

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To Bill

and

to Kathleen,

for her birthday

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LIST OF CONCRETE CATEGORIES

<u>Ab</u>	The category of Abelian groups and group homomorphisms
<u>AbMon</u>	The category of Abelian monoids and monoid homomorphisms
<u>CabMon</u>	The category of cancellative Abelian monoids and monoid homomorphisms
<u>CompT</u>$_2$	The category of compact Hausdorff spaces and continuous functions
<u>CompRegT</u>$_1$	The category of all completely regular T_1 spaces and continuous functions
<u>Field</u>	The category of all fields and field homomorphisms
<u>Grp</u>	The category of all groups and group homomorphisms
<u>Haus</u>	The category of all Hausdorff spaces and continuous functions
<u>Ind</u>	The category of all indiscrete spaces and continuous functions
<u>Mon</u>	The category of all monoids and monoid homomorphisms
<u>POS</u>	The category of all partially ordered sets and order-preserving functions
<u>RComp</u>	The category of all realcompact spaces and continuous functions
<u>Set</u>	The category of all sets and functions
<u>Sgp</u>	The category of all semigroups and semigroup homomorphisms
<u>Top</u>	The category of all topological spaces and continuous functions
<u>Top</u>$_1$	The category of all T_1 spaces and continuous functions

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A recent definition for embedding morphisms in concrete categories is examined. Several types of sources are defined for which the source morphisms in the aggregate exhibit the character of single morphisms with 'embedding-like' properties. A theorem giving necessary and sufficient conditions for embedding an object into a categorical product of objects is proven for a variety of 'embedding-like' morphisms. The concept of embeddable objects is examined, and a definition is developed for $\mathcal{E}$-regular and $\mathcal{E}$-compact objects in concrete categories. In consequence, several characterization theorems for epireflective subcategories and epireflective hulls, which previously had been proven only for certain categories of topological spaces, are extended to a variety of "reasonable" concrete categories.

Baayen's generalizations of de Groot's results, on the existence of universal linearizations for monoids of endomorphisms

on completely regular T_1-spaces, are extended. For every isomorphism-closed, left-cancellative class $\mathcal{M}$ of morphisms in a category with countable products, every endomorphism on an object in the category is shown to be a restriction, relative to $\mathcal{M}$, of a coordinate-immuting endomorphism on a power of an object in the category. Under the same conditions, an automorphism in the category is shown to be the restriction of a coordinate-permuting automorphism. Furthermore, simultaneous $\mathcal{M}$-linearizations are shown to exist for certain monoids of endomorphisms on objects in the category, and universal $\mathcal{M}$-linearizations are shown to exist for every endomorphism in certain subcategories of the category.

INTRODUCTION

The "mathematical-man-in-the-street," when asked what he would consider an embedding morphism to be in any concrete category, would probably reply, if he replied at all, that it would have to be injective on the underlying sets and would have to preserve the structure of the embedded object; i.e., it would have to be an actual subsystem embedding. It has been somewhat difficult to find a categorical definition for a morphism that does just that. Several types of morphisms have been defined which act as subsystem embeddings in some categories but are too weak or too strong in other categories. Monomorphisms in algebraic categories usually act as embeddings, but they are not embeddings in the topological sense in Top; extremal monomorphisms are precisely the topological embeddings in Top, but they are too restrictive in Haus where they are homeomorphisms onto closed subspaces, and consequently both sections and regular monomorphisms are too restrictive in Haus. Within the last year, Herrlich and Strecker [7] developed the definition for concrete embeddings in a concrete category, which they found to be precisely the topological embeddings in Top and Haus and to be precisely the monomorphisms in algebraic categories (1.2.2). In Chapter 1, §1.5, we shall begin to examine the concept of concrete embeddings; in Chapter 3, we shall begin to exploit it. Several new results will be proved in §1.5. We will find that concrete embeddings are always injective on the underlying sets, and that concrete embeddings tend to have stable hereditary properties; i.e., that concrete embeddings in many subcategories tend to be concrete embeddings in the larger category, and conversely.

Frequently, when dealing with concrete embeddings, we restrict our attention to concrete categories $(\mathcal{C}, \mathcal{U})$ in which the faithful functor $\mathcal{U}: \mathcal{C} \to \underline{\mathrm{Set}}$ preserves products or monomorphisms. This restriction is fairly weak; in most ordinary concrete categories, the functor $\mathcal{U}$ has a left-adjoint and hence preserves all limits (and hence all monomorphisms too).

In Chapter 2, we will develop the notion of sources. A source in a category $\mathcal{C}$ is a $\mathcal{C}$-object X together with a family of $\mathcal{C}$-morphisms having X as their common domain. A categorical product $(\prod_{i \in I} E_i, \pi_i)$ is an example of a source which, in fact, exhibits a "mono-like" property; i.e., if f and g are morphisms in the category such that $\pi_i \cdot f = \pi_i \cdot g$ for each $i \in I$, then $f = g$. We call a source, whose morphisms acting in concert exhibit the property of a single monomorphism, a mono source. Herrlich and Strecker [7] have developed definitions for mono sources and extremal mono sources. In Chapter 2, we shall define several new types of sources (called, collectively, $\mathcal{M}$-sources) whose morphisms in the aggregate exhibit the properties of special types of "embedding-like" morphisms ($\mathcal{M}$-morphisms).

In topology, a long-established method for producing mappings into topological products has been by use of families of morphisms into the coordinate spaces. Tychonoff's well-known result, that for any infinite cardinal number k, a completely regular T_1 space X of weight $\leq k$ can be topologically embedded into the topological product $[0,1]^k$ of unit intervals, made use of the family $\mathcal{C}[X,[0,1]]$ of all continuous functions from X into the unit interval [0,1].

His result stimulated several mathematicians to investigate this general procedure for topologically embedding a space into a product of spaces. In 1956, Mrówka [13] published a theorem giving necessary and sufficient conditions for a function to be a topological embedding of a space into a topological power of another space. More recently [16], he expanded this result to include embeddings of a space into products of other spaces, as well as characterizing topological embeddings onto closed subspaces of products of Hausdorff spaces. From our vantage point in category theory, we can see that these theorems concerned sources: that certain conditions on the source $(X, \mathcal{F})$ where $\mathcal{F}$ is a family of continuous maps from the space X into coordinate spaces of a topological product are necessary and sufficient to guarantee the existence of a homeomorphism from X onto a subspace of the product. In Chapter 3, we will prove a similar embedding theorem for categories (3.1.1) for a variety of "embedding-type" morphisms.

In 1958, Engelking and Mrówka [3] began to develop the concepts of E-regular and E-compact spaces in response to two intriguing questions: (1) for a given space E, when can a space X be topologically embedded into some topological power of E? (When is X an E-completely regular space?), and (2) for a given Hausdorff space E, when is a space X homeomorphic to a closed subspace of some topological power of E? (When is X an E-compact space?) Later Herrlich [4] expanded these investigations to include the concepts $\mathcal{E}$-regular (respectively, $\mathcal{E}$-compact) spaces for collections $\mathcal{E}$ of spaces (respectively, Hausdorff spaces), initiating the use of a category-theoretic approach.

In §3.2, we shall characterize objects which are "embeddable" in products of other objects. In particular for a concrete category $(\mathcal{C},\mathcal{U})$ with a subcategory $\mathcal{E}$, we will define a $\mathcal{C}$-object X to be $\mathcal{E}$-regular in $\mathcal{C}$ (respectively, $\mathcal{E}$-compact in $\mathcal{C}$) provided that there exists a concrete embedding (respectively, an extremal concrete embedding) from X into the object part of a product of $\mathcal{E}$-objects. For the first time, the concepts of $\mathcal{E}$-regular and $\mathcal{E}$-compact objects can be applied to categories other than <u>Top</u> or <u>Haus</u>.

The most interesting applications of these concepts will be in the realm of epireflective subcategories. Recall that a subcategory $\mathcal{A}$ is epireflective in a category $\mathcal{C}$ provided that for every $\mathcal{C}$-object X there exists a pair $(r_{\mathcal{A}}, X_{\mathcal{A}})$, called the $\mathcal{A}$-epireflection of X, where $X_{\mathcal{A}}$ is an $\mathcal{A}$-object and $r_{\mathcal{A}}:X \to X_{\mathcal{A}}$ is an epimorphism with a maximal extension property for $\mathcal{A}$. (In this paper, we will say that $r_{\mathcal{A}}$ is "$\mathcal{A}$-extendable" provided that for every morphism $g:X \to A$ for some $\mathcal{A}$-object A, there exists a morphism $g^*:X_{\mathcal{A}} \to A$ such that $g^* \cdot r_{\mathcal{A}} = g$.) Herrlich and Strecker [8] have characterized epireflective subcategories in the following fashion: in a complete, well-powered and cowell-powered category $\mathcal{C}$, with a full, replete subcategory $\mathcal{A}$, $\mathcal{A}$ is epireflective in $\mathcal{C}$ if and only if it is closed under the formation of extremal subobjects and products in $\mathcal{C}$. This theorem gives us a large array of epireflective subcategories. The actual construction of the epireflection can be quite difficult. Usually the construction takes one of two forms: (1) "factoring out" the elements in the object with undesirable characteristics, e.g., the Abelianization of a group G by factoring

out the commutator subgroup G_c of G ($a:G \to G/G_c$), or the Hausdorffication of a topological space X by finding the appropriate equivalence relation R so that X/R is a Hausdorff space with the "universal" mapping property, or (2) constructing a "larger" object with the desired structure and finding an epimorphism from the object to the "larger" object, as in the Stone-Čech construction, $\beta:X \to \beta X$, which gives a $\underline{\mathrm{CompT}_2}$-epireflection for a Hausdorff space X and which gives a topological embedding, as well, when X is a completely regular T_1 space, or as in the construction of the Grothendieck group K(M) for an Abelian monoid, $k:M \to K(M)$, for which k will be an injection, as well, when M is a cancellative Abelian monoid. It has always been interesting to note that in certain cases, the second method described above results in epireflections that are actual subsystem embeddings. Both methods of constructing epireflections have been used for both topological and algebraic categories; yet, until now, there has been no general categorical way to differentiate between these two methods and to determine for what objects in the category the epireflection epimorphism will be an actual subsystem embedding. For the category Haus, Herrlich and Van der Slot [10] were able to show that a full, replete subcategory $\mathfrak{A}$ is epireflective if and only if for every $\mathfrak{A}$-regular object X in Haus there exists an epimorphism $r:X \to A$ to some $\mathfrak{A}$-object A such that r is an $\mathfrak{A}$-extendable topological embedding. In §2.4, we will extend this result to one for "reasonable" concrete categories, and hence will be able to add a new characterization for epireflective subcategories in those categories. Furthermore, we will extend several other theorems (which Herrlich [5] proved for the

category Haus) which will characterize $\mathfrak{A}$-regular and $\mathfrak{A}$-compact objects in "reasonable" concrete categories.

Tychonoff's embedding result, mentioned above, gave rise to another area of investigation. De Groot [2] proved that for a given infinite cardinal number k, there exists a monoid F of endomorphisms on $[0,1]^k$ which "universally linearizes" any monoid S of at most k endomorphisms on any completely regular T_1 space X of weight $\leq$ k. Baayen [1] generalized this result to obtain several theorems in categorical terms, restricting his considerations to monomorphisms for general categories and topological embeddings for Top, which he had to consider separately.

In Chapter 4, we will extend and update Baayen's theorems, but, in particular, we will change the emphasis of the investigation. We will not be searching for universal objects in categories as Baayen did; instead, we will obtain quite general categorical linearization theorems for endomorphisms (and automorphisms) on objects in a category. Considering categories in which only certain products are required to exist, we will find that endomorphisms are actually restrictions of morphisms of an almost trivial-seeming linear character (i.e., morphisms that essentially act only on the coordinates of a power of an object, serving to "switch or collapse" these coordinates).

We will find that in categories with countable products, every endomorphism in the category can be 'linearized', i.e., extended to a coordinate immuting morphism (4.2.10), and that any automorphism can be considered as the 'restriction' of a coordinate permuting automorphism on a power. Furthermore, we will find that certain monoids of

endomorphisms, as well as groups of automorphisms, on an object in the category can be simultaneously linearized (4.2.4). And finally, we will find that for certain subcategories in a category with infinite products, universal linearizations may exist for every endomorphism in the subcategory (4.3.2).

1. PRELIMINARIES

The purpose of this chapter is to list most of the basic definitions and theorems that will be used in the following chapters. Proofs will be included only for new results, which will be found in §1.5. Terms from category theory not specifically defined here can be found in [7], as well as the proofs for those theorems and examples that are stated here without proof or reference. Algebraic terms can be found in [12]; topological terms can be found in [11].

§1.1 Products and Limits

DEFINITION 1.1.1 A <u>family</u> $(A_i)_{i\varepsilon I}$ <u>indexed by</u> I is a function A with domain I. For each $i\varepsilon I$, A(i) is usually written A_i. Occasionally when I is unimportant or understood, the family $(A_i)_{i\varepsilon I}$ is written (A_i).

DEFINITION 1.1.2 Let $\mathcal{C}$ be a category. A <u>product in</u> $\mathcal{C}$ of a set-indexed family $(A_i)_{i\varepsilon I}$ of $\mathcal{C}$-objects is a pair (usually denoted by) $(\prod_{i\varepsilon I} A_i, (\pi_i)_{i\varepsilon I})$ satisfying the following three properties:

(1) $\prod_{i\varepsilon I} A_i$ is a $\mathcal{C}$-object

(2) for each $j\varepsilon I$, $\pi_j: \prod_{i\varepsilon I} A_i \to A_j$ is a $\mathcal{C}$-morphism (called the <u>projection from</u> $\prod_{i\varepsilon I} A_i$ <u>to</u> A_j).

(3) for each pair $(C,(f_i)_{i\varepsilon I})$, where C is a $\mathcal{C}$-object and for each $j\varepsilon I$, $f_j: C \to A_j$ is a $\mathcal{C}$-morphism, there exists a unique induced $\mathcal{C}$-morphism (usually denoted by) $\langle f_i \rangle: C \to \prod_{i\varepsilon I} A_i$ such that for each $j\varepsilon I$, the triangle

$$\begin{tikzcd} C \arrow[rr, dashed, "\langle f_i \rangle"] \arrow[drr, "f_j"'] & & \prod_{i\varepsilon I} A_i \arrow[d, "\pi_j"] \\ & & A_j \end{tikzcd}$$

commutes.

For notational convenience we may sometimes write $(\Pi A_i, \pi_i)$ for the product. Also when $A_i = B$ for each $i\varepsilon I$, the product $(\Pi A_i, \pi_i)$ is usually written (B^I, π_i) and is called the I'th power of B in $\mathcal{C}$. A category $\mathcal{C}$ is said to <u>have products</u> provided that for every set I, each family of $\mathcal{C}$-objects indexed by I has a product in $\mathcal{C}$. The following categories have products: <u>Set</u>, <u>Grp</u>, <u>Top</u>, <u>Haus</u>, <u>Top_1</u>, <u>$CompT_2$</u>, <u>$CompRegT_1$</u>, <u>Mon</u>, <u>Sgp</u>.

THEOREM 1.1.3 (Iteration of Products) Let $(K_i)_{i\varepsilon I}$ be a pairwise disjoint set-indexed family of sets. Suppose that for each $i\varepsilon I$, $(P^i, (\pi^i_k)_{k\varepsilon K_i})$ is the product of a set-indexed family $(X_k)_{k\varepsilon K_i}$ of $\mathcal{C}$-objects and that (P, π_i) is the product of the family $(P^i)_{i\varepsilon I}$. Then $(P, (\pi^i_k \cdot \pi_i)_{i\varepsilon I, k\varepsilon K_i})$ is the product of $(X_k)_{k\varepsilon \bigcup_I K_i}$ up to a natural isomorphism.

PROPOSITION 1.1.4. If $(\Pi A_i, \pi_i)$ and $(\Pi B_i, \rho_i)$ are products of the families $(A_i)_{i\varepsilon I}$ and $(B_i)_{i\varepsilon I}$, respectively, and if for each $i\varepsilon I$ there is a morphism $A_i \xrightarrow{f_i} B_i$, then there exists a unique morphism (usually denoted by Πf_i) which makes the diagram

$$\begin{tikzcd} \Pi A_i \arrow[rr, dashed, "\pi f_i"] \arrow[d, "\pi_j"'] & & \Pi B_i \arrow[d, "\rho_j"] \\ B_i \arrow[rr, "f_j"] & & B_j \end{tikzcd}$$

commute for each $j \varepsilon I$.

DEFINITION 1.1.5. The morphism Πf_i of Proposition 1.1.4 is called the <u>product of the morphisms</u> $(f_i)_{i \varepsilon I}$.

Products are just a special type of limit which will be defined next.

DEFINITION 1.1.6. Let I and $\mathcal{C}$ be categories and let $D: I \to \mathcal{C}$ be a functor.

(1) Then a pair $(L, (f_i)_{i \varepsilon Ob(I)})$ is called a <u>natural source</u> for D in $\mathcal{C}$ provided that L is a $\mathcal{C}$-object, and for each $i \varepsilon Ob(I)$, $f_i: L \to D_i$ is a $\mathcal{C}$-morphism, and for all $i, j \varepsilon Ob(I)$ and I-morphisms $m: i \to j$, the triangle

$$\begin{tikzcd} & D(i) \arrow[dd, "D(m)"] \\ L \arrow[ur, "f_i"] \arrow[dr, "f_j"'] & \\ & D(j) \end{tikzcd}$$

commutes.

(2) A natural source $(L, f_i)_{i \varepsilon Ob(I)})$ for D in $\mathcal{C}$ is called a <u>limit for D in</u> $\mathcal{C}$ provided that if $(\hat{L}, (\hat{f}_i)_{i \varepsilon Ob(I)})$ is any other natural source for D in $\mathcal{C}$, then there is a unique morphism $h: \hat{L} \to L$ such that

for each $j \varepsilon Ob(I)$, the triangle

$$\begin{tikzcd}
\hat{L} \arrow[dr, "\hat{f}_j"] \arrow[dd, dashed, "h"'] & \\
& D(j) \\
L \arrow[ur, "f_j"'] &
\end{tikzcd}$$

commutes.

Some well-known examples of limits are: products, terminal objects, equalizers, pullbacks, and intersections.

DEFINITION 1.1.7. A category $\mathcal{C}$ is said to be _complete_ provided that for every small category I (i.e., where Ob(I) is a set), every functor $D:I \to \mathcal{C}$ has a limit.

(The categories _Set_, _Grp_, _Mon_, _Top_, _Haus_ and $\underline{CompT_2}$ are examples of complete categories.)

§1.2 Special Categories

DEFINITION 1.2.1. A category $\mathcal{C}$ is said to be _well-powered_ provided that every $\mathcal{C}$-object has a representative class of subobjects which is a _set_.

Dual notion: _cowell-powered_

The categories _Set_, _Grp_, _Top_, _Mon_, _Haus_ and $\underline{CompT_2}$ are well-powered and cowell-powered.

DEFINITION 1.2.2

(1) A _concrete category_ is a pair $(\mathcal{C}, \mathcal{U})$ where $\mathcal{C}$ is a category and $\mathcal{U} : \mathcal{C} \to$ _Set_ is a faithful functor.

(2) A concrete category $(\mathcal{A}, \mathcal{U})$ is called algebraic provided that it satisfies the following three conditions:

(a) $\mathcal{A}$ has coequalizers.

(b) $\mathcal{U}$ has a left-adjoint.

(c) $\mathcal{U}$ preserves and reflects regular epimorphisms.

Examples of concrete categories include Set, Grp, Top, Haus, POS, Field and Mon. Top, Haus, POS and Field are not algebraic categories; however, Set, Grp, Mon and CompT_2 are algebraic.

§1.3 Special Morphisms in General Categories

DEFINITION 1.3.1. Let $\mathcal{C}$ be a category.

(1) A $\mathcal{C}$-morphism m is called a strong morphism provided that whenever $m \cdot k = g \cdot e$ for some $\mathcal{C}$-morphisms k and g and some $\mathcal{C}$-epimorphism e, there exists a morphism h such that the diagram

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "k"'] & \cdot \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$

commutes.

(2) A $\mathcal{C}$-morphism m is called a strong monomorphism when it is both a strong morphism and a monomorphism.

(3) A $\mathcal{C}$-morphism f is called an extremal morphism provided that whenever $f = g \cdot e$, where g is a $\mathcal{C}$-morphism and e is a $\mathcal{C}$-epimorphism, e must be a $\mathcal{C}$-isomorphism.

(4) A $\mathcal{C}$-morphism is called an extremal monomorphism provided

that it is both an extremal morphism and a monomorphism.

PROPOSITION 1.3.2. Let $\mathcal{C}$ be a category.

(a) The product of strong (mono) morphisms is a strong (mono) morphism.

(b) If f and g are strong (mono)morphisms and $f \cdot g$ is a morphism, then $f \cdot g$ is a strong (mono) morphism.

Thus we say that the class of strong morphisms (respectively, strong monomorphisms) is closed under products in $\mathcal{C}$ (1.3.2(a)) and closed under composition in $\mathcal{C}$ (1.3.2(b)). In general, this is not true for the class of extremal morphisms in $\mathcal{C}$ or for the class of extremal monomorphisms in $\mathcal{C}$.

Extremal monomorphisms are particularly interesting. The following definitions and propositions will give us sufficient conditions for which the class of extremal monomorphisms is closed under products and composition.

DEFINITION 1.3.3. Let $\mathcal{C}$ be a category, $\mathcal{M}$ be a class of monomorphisms closed under composition with isomorphisms, and $\mathcal{E}$ be a class of epimorphisms closed under composition with isomorphisms.

(1) Let f be a morphism in $\mathcal{C}$. A factorization $f = m \cdot e$ where e and m are morphisms in $\mathcal{C}$ is called an $(\mathcal{E},\mathcal{M})$ <u>factorization of</u> f provided that $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

(2) An $(\mathcal{E},\mathcal{M})$ factorization of f, $f = m \cdot e$, is said to be <u>unique</u> provided that whenever $f = \hat{m} \cdot \hat{e}$ is another $(\mathcal{E},\mathcal{M})$ factorization of f, there is an isomorphism h such that the diagram

$$\begin{tikzcd} & \bullet \arrow[dr, "\hat{m}"] \arrow[dd, dashed, "h"] & \\ \bullet \arrow[ur, "\hat{e}"] \arrow[dr, "e"'] & & \bullet \\ & \bullet \arrow[ur, "m"'] & \end{tikzcd}$$

commutes.

(3) A category $\mathcal{C}$ is called an $(\mathcal{L},\mathcal{M})$ <u>category</u> provided that $\mathcal{L}$ and $\mathcal{M}$ are closed under composition and every morphism in $\mathcal{C}$ has a unique $(\mathcal{L},\mathcal{M})$ factorization.

(4) A category $\mathcal{C}$ is said to have the $(\mathcal{L},\mathcal{M})$ <u>diagonalization property</u> provided that for every commutative square in $\mathcal{C}$

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

with $e\varepsilon\mathcal{L}$ and $m\varepsilon\mathcal{M}$ there exists a morphism k which makes the diagram

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \arrow[dl, dashed, "k"'] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

commute.

PROPOSITION 1.3.4. Every $(\mathcal{L},\mathcal{M})$ category has the $(\mathcal{L},\mathcal{M})$ diagonalization property.

THEOREM 1.3.5. Every complete, well-powered category $\mathcal{C}$ is an (epi, extremal mono) category. Furthermore the class of extremal monomorphisms in $\mathcal{C}$ is closed under products.

§1.4 Epireflections

DEFINITION 1.4.1. Let $\mathcal{C}$ be a category and let $\mathcal{A}$ be a subcategory of $\mathcal{C}$.

(1) $\mathcal{A}$ is a <u>full subcategory</u> of $\mathcal{C}$ provided that whenever $f:A \to B$ is a $\mathcal{C}$-morphism, and A and B are $\mathcal{A}$-objects, it follows that f is an $\mathcal{A}$-morphism.

(2) $\mathcal{A}$ is a <u>replete subcategory of</u> $\mathcal{C}$ provided that whenever $f:A \to B$ is an $\mathcal{C}$-isomorphism and A is an $\mathcal{A}$-object, it follows that B is an $\mathcal{A}$-object.

DEFINITION 1.4.2. Let $\mathcal{C}$ be a category and let $\mathcal{A}$ be a subcategory of $\mathcal{C}$.

(1) A morphism $f:X \to Y$ is called $\mathcal{A}$-<u>extendable</u> provided that for each $\mathcal{A}$-object A and each morphism $g:X \to A$ there exists a morphism $g^*:Y \to A$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, dashed, "g^*"] \\ & A & \end{tikzcd}$$

commutes.

(2) Let X be a $\mathcal{C}$-object. The pair $(r_{\mathfrak{A}}, X_{\mathfrak{A}})$ is called an $\mathfrak{A}$-<u>epireflection for</u> X provided that $X_{\mathfrak{A}}$ is an $\mathfrak{A}$-object and $r_{\mathfrak{A}}: X \to X_{\mathfrak{A}}$ is an $\mathfrak{A}$-extendable epimorphism.

(3) $\mathfrak{A}$ is called an epireflective subcategory of $\mathcal{C}$ provided that for every $\mathcal{C}$-object X, there exists an $\mathfrak{A}$-epireflection for X.

THEOREM 1.4.3 (Characterization of Epireflective Subcategories) (Herrlich and Strecker [8]). Let $\mathcal{C}$ be a complete, well-powered and cowell-powered category and let $\mathfrak{A}$ be a full, replete subcategory of $\mathcal{C}$. Then the following statements are equivalent:

(a) $\mathfrak{A}$ is an epireflective subcategory of $\mathcal{C}$.

(b) $\mathfrak{A}$ is closed under the formation of products and extremal subobjects in $\mathcal{C}$.

PROPOSITION 1.4.4. Let $\mathcal{C}$ be a complete, well-powered and cowell-powered category and let $\mathfrak{A}$ be any full, replete subcategory of $\mathcal{C}$. Then there exists a smallest epireflective subcategory $\mathcal{C}(\mathfrak{A})$ of $\mathcal{C}$ which contains $\mathfrak{A}$.

DEFINITION 1.4.5. Let $\mathcal{C}$ be a category and let $\mathfrak{A}$ be a subcategory of $\mathcal{C}$. The epireflective hull $\mathcal{C}(\mathfrak{A})$ of $\mathfrak{A}$ in $\mathcal{C}$ (if it exists) is the smallest epireflective subcategory of $\mathcal{C}$, containing $\mathfrak{A}$.

THEOREM 1.4.6 (Characterization of Epireflective Hulls) (Herrlich [6]). Let $\mathcal{C}$ be a complete, well-powered and cowell-powered category and let $\mathfrak{A}$ be any full, replete subcategory of $\mathcal{C}$. Let X be a $\mathcal{C}$-object. Then the following statements are equivalent:

(a) X is a $\mathcal{C}(\mathfrak{A})$-object

(b) X is an extremal subobject of a product of $\mathfrak{A}$-objects

(c) Each $\mathfrak{A}$-extendable epimorphism is $\{X\}$-extendable

(d) Each $\mathcal{A}$-extendable epimorphism $f:X \to Y$ is an isomorphism

(e) Each $\mathcal{A}$-extendable morphism $f:X \to Y$ is an extremal monomorphism.

§1.5 Concrete Embeddings

In this section, we will define concrete embeddings and will determine a few of their properties.

DEFINITION 1.5.1. Let $(\mathcal{C},\mathcal{U})$ be a concrete category.

(1) A morphism $f:X \to Y$ is called a <u>concrete embedding</u> provided that it is a monomorphism and whenever there is a morphism $g:Z \to Y$ for which there exists a function $h:\mathcal{U}(Z) \to \mathcal{U}(X)$ such that the diagram

$$\begin{tikzcd}
\mathcal{U}(Z) \arrow[d, "h"'] \arrow[dr, "\mathcal{U}(g)"] & \\
\mathcal{U}(X) \arrow[r, "\mathcal{U}(f)"'] & \mathcal{U}(Y)
\end{tikzcd}$$

commutes, there exists a morphism $\bar{h}:Z \to X$ such that $\mathcal{U}(\bar{h}) = h$.

(2) A morphism $f:X \to Y$ is called an <u>extremal</u> (respectively <u>strong</u>) <u>concrete embedding</u> provided that it is both a concrete embedding and an extremal (strong) morphism.

PROPOSITION 1.5.2. Let $(\mathcal{C},\mathcal{U})$ be a concrete category.

(1) The class of all concrete embeddings in $\mathcal{C}$ is closed under composition.

(2) If $\mathcal{U}$ preserves monomorphisms, the class of all concrete embeddings in $\mathcal{C}$ is closed under products, intersections and pullbacks.

PROPOSITION 1.5.3. Let $(\mathcal{C},\mathcal{U})$ be a concrete category that is complete and well-powered for which $\mathcal{U}$ preserves monomorphisms. Then every extremal monomorphism is a concrete embedding, hence an extremal concrete embedding.

COROLLARY 1.5.4. Let $(\mathcal{C},\mathcal{U})$ be a concrete category that is complete and well-powered for which $\mathcal{U}$ preserves monomorphisms. Then $(\mathcal{C},\mathcal{U})$ is an (epi, extremal concrete embedding) category.

In some concrete categories, monomorphisms are not always injective on the underlying sets. We next show that concrete embeddings do have the desired property of injectivity.

PROPOSITION 1.5.5. Let $(\mathcal{C},\mathcal{U})$ be a concrete category. Then every concrete embedding in $\mathcal{C}$ must be injective on the underlying sets.

PROOF: Suppose that $f:X \to Y$ is a concrete embedding, and hence a monomorphism (1.5.1), but that $\mathcal{U}(f)$ is not injective on the underlying sets. Then there must exist $a,b\in\mathcal{U}(X)$, $a \neq b$ such that $\mathcal{U}(f)(a)=\mathcal{U}(f)(b)$.

Define $h_a: \mathcal{U}(X) \to \mathcal{U}(X)$ such that $h_a(x) = \begin{cases} a & \text{for } x=a,\ x=b \\ x & \text{for all } x\in X, \text{ for which } a \neq x \neq b \end{cases}$

and define $h_b: \mathcal{U}(X) \to \mathcal{U}(X)$ such that $h_b(x) = \begin{cases} b & \text{for } x=a,\ x=b \\ x & \text{for all } x\in X \text{ for which } a \neq x \neq b. \end{cases}$

Then both h_a and h_b make the diagram

$$\begin{tikzcd} \mathcal{U}(X) \arrow[d, shift right, "h_a"'] \arrow[d, shift left, "h_b"] \arrow[dr, "\mathcal{U}(f)"] & \\ \mathcal{U}(X) \arrow[r, "\mathcal{U}(f)"'] & \mathcal{U}(Y) \end{tikzcd}$$

commute. Thus since f is a concrete embedding, there must exist $\overline{h}_a : X \to X$ such that $\mathcal{U}(\overline{h}_a) = h_a$ and $\overline{h}_b : X \to X$ such that $\mathcal{U}(\overline{h}_b) = h_b$ (1.5.1). But $\mathcal{U}$ is a faithful functor; hence it reflects commutative triangles. Thus $f \cdot \overline{h}_a = f = f \cdot \overline{h}_b$, so $\overline{h}_a = \overline{h}_b$ since f is a monomorphism. But this implies that $h_a = h_b$--a contradiction.

It is known ([7]) that concrete embeddings in Top are precisely the topological embeddings, and that concrete embeddings in algebraic categories are precisely the monomorphisms. We will show that these results can be used to characterize concrete embeddings in many additional categories.

PROPOSITION 1.5.6. Let $(\mathcal{C}, \mathcal{U})$ be a concrete category, let $\mathcal{A}$ be any full subcategory of $\mathcal{C}$ and let $f : X \to Y$ be an $\mathcal{A}$-morphism. If f is a concrete embedding in $\mathcal{C}$, then it is a concrete embedding in $\mathcal{A}$.

PROOF: Let $f : X \to Y$ be a concrete embedding in $\mathcal{C}$. Then it is injective on the underlying sets (1.5.5), hence it is a monomorphism in $\mathcal{A}$. Let $g : Z \to Y$ be an $\mathcal{A}$-morphism such that for some function $h : \mathcal{U}(Z) \to \mathcal{U}(X)$, $\mathcal{U}(f) \cdot h = \mathcal{U}(g)$. But g is also a $\mathcal{C}$-morphism and f is a concrete embedding in $\mathcal{C}$; hence, there exists a $\mathcal{C}$-morphism

$\overline{h}:Z \to X$ such that $\mathcal{U}(\overline{h}) = h$. (1.5.1). Since $\mathfrak{A}$ is full, $\overline{h}:Z \to X$ is also an $\mathfrak{A}$-morphism.

PROPOSITION 1.5.7. Let $\mathcal{C}$ be an algebraic category, let $\mathfrak{A}$ be any subcategory of $\mathcal{C}$, and let $f:A \to B$ be an $\mathfrak{A}$-morphism. Then if f is a concrete embedding in $\mathfrak{A}$, it is a concrete embedding in $\mathcal{C}$. Furthermore if $\mathfrak{A}$ is a full subcategory of $\mathcal{C}$, then f is a concrete embedding in $\mathcal{C}$ if and only if it is a concrete embedding in $\mathfrak{A}$.

PROOF: Suppose f is a concrete embedding in $\mathfrak{A}$. Then it is injective on the underlying sets (1.5.5), and consequently it is a monomorphism in $\mathcal{C}$, thus a concrete embedding in $\mathcal{C}$.

The remainder of the proof follows directly from Proposition 1.5.6.

PROPOSITION 1.5.8. Let $(\mathcal{C},\mathcal{U})$ be a concrete category that is complete and well-powered and for which $\mathcal{U}$ preserves epimorphisms and monomorphisms. Let $\mathfrak{A}$ be a full, subcategory of $\mathcal{C}$ that is closed under the formation of extremal subobjects in $\mathcal{C}$, and let $f:A \to B$ be an $\mathfrak{A}$-morphism. Then the following statements are equivalent:

(a) f is a concrete embedding in $\mathcal{C}$.

(b) f is a concrete embedding in $\mathfrak{A}$.

PROOF: <u>(a) $\Rightarrow$ (b)</u>: The proof follows directly from Proposition 1.5.6, since $\mathfrak{A}$ is a full subcategory of $\mathcal{C}$.

<u>(b) $\Rightarrow$ (a)</u>: Suppose f is a concrete embedding in $\mathfrak{A}$. Then, since $\mathcal{C}$ is a complete well-powered category, there exists a unique (epi, extremal mono) factorization of f, $f = m \cdot e$ (1.3.5). Let L denote the domain of m.

$$\begin{tikzcd} & L \arrow[dr, "m"] & \\ A \arrow[ur, "e"] \arrow[rr, "f"'] & & B \end{tikzcd}$$

By hypothesis, $\mathcal{U}$ preserves epimorphisms; thus $\mathcal{U}(e):\mathcal{U}(A) \to \mathcal{U}(L)$ is surjective. But $\mathcal{U}(f)$ is injective (1.5.5); so that $\mathcal{U}(e)$ must be injective; hence bijective. Consequently there is a function $h: \mathcal{U}(L) \to \mathcal{U}(A)$ such that $\mathcal{U}(e)\cdot h = 1_{\mathcal{U}(L)}$ and $h\cdot \mathcal{U}(e) = 1_{\mathcal{U}(A)}$. Thus $\mathcal{U}(f)\cdot h = \mathcal{U}(m)$.

But (L,m) is an extremal subobject of B which is an $\mathfrak{A}$-object. Hence by hypothesis, L is an $\mathfrak{A}$-object, so m is an $\mathfrak{A}$-morphism. Thus since f is a concrete embedding in $\mathfrak{A}$, there exists an $\overline{h}:L \to A$ such that $\mathcal{U}(\overline{h}) = h$. Thus $\overline{h}\cdot e:A \to A$ and $\mathcal{U}(\overline{h}\cdot e) = h\cdot \mathcal{U}(e) = 1_{\mathcal{U}(A)} = \mathcal{U}(1_A)$. Hence since $\mathcal{U}$ is faithful, $\overline{h}\cdot e = 1_A$. Thus e is a section (and an epimorphism); hence it is an isomorphism. Thus f is an extremal monomorphism in $\mathcal{C}$ (1.3.1); hence it is a concrete embedding in $\mathcal{C}$ (1.5.3).

COROLLARY 1.5.9. In any full, hereditary subcategory $\mathfrak{A}$ of Top, the concrete embeddings are precisely the topological embeddings.

PROOF: The epimorphisms in Top are precisely the surjective continuous functions.

2. SOURCES

In this chapter we will investigate the action of sources; a source being an object in a category together with a family of morphisms having the object as a common domain. We will see that in several special cases the morphisms of a source will act together to exhibit certain "mono-like" properties--properties which may not belong to any of the morphisms individually. The definitions and initial results on mono sources and extremal mono sources were developed by Herrlich and Strecker [7].

§2.1 Sources in General Categories

DEFINITION 2.1.1. Let $\mathcal{C}$ be a category.

(1) The pair $(A,(f_i)_{i\varepsilon I})$ is called a source provided that A is a $\mathcal{C}$-object and $(f_i)_{i\varepsilon I}$ is a family of $\mathcal{C}$-morphisms, each with domain A. Note that for notational convenience, we will usually write (A,f_i) for a source when the indexing class I is understood or unimportant.

(2) A source (A,f_i) is called a mono source provided that for any pair of $\mathcal{C}$-morphisms h and k such that $f_i \cdot h = f_i \cdot k$ for all $i \varepsilon I$, it follows that $h = k$; i.e., provided that the family $(f_i)_{i\varepsilon I}$ is simultaneously left-cancellable.

(3) A source (A,f_i) is called an extremal source provided that for each source (B,g_i) and each $\mathcal{C}$-epimorphism $e:A \to B$ such that for each i, the diagram

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "f_i"'] & & B \arrow[dl, "g_i"] \\ & C_i & \end{tikzcd}$$

commutes, e must be an isomorphism.

(4) A source (A,f_i) is called an <u>extremal mono source</u> provided that it is both an extremal source and a mono source.

(5) A source (A,f_i) is called a <u>strong source</u> provided that for each source (B,g_i) and $\mathcal{C}$-morphisms e and k, where e is a $\mathcal{C}$-epimorphism and $g_i \cdot e = f_i \cdot k$ for each $i \varepsilon I$, there exists a $\mathcal{C}$-morphism h such that the diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "k"'] & B \arrow[d, "g_i"] \arrow[dl, dashed, "h"'] \\ A \arrow[r, "f_i"'] & C_i \end{tikzcd}$$

commutes for each $i \varepsilon I$.

(6) A source (A,f_i) is called a <u>strong mono source</u> provided that it is a strong source and a mono source.

In the following paragraphs we will examine some fundamental examples of these sources. First, however, we shall determine their relative strengths.

PROPOSITION 2.1.2. Let $\mathcal{C}$ be a category and let (A,f_i) be a source in $\mathcal{C}$. Then each of the following statements implies the statement below it.

(a) (A,f_i) is a strong mono source (resp., strong source).

(b) (A,f_i) is an extremal mono source (resp., extremal source).

(c) (A,f_i) is a mono source (resp., source).

PROOF: <u>(a) $\Rightarrow$ (b)</u>: Let (A,f_i) be a strong source in $\mathcal{C}$. Suppose there exists a family $(g_i)_{i\varepsilon I}$ of morphisms in $\mathcal{C}$ and an epimorphism e in $\mathcal{C}$ such that $f_i = g_i \cdot e$ for all $i\varepsilon I$. Then for all $i\varepsilon I$, $f_i \cdot 1_A = g_i \cdot e$. Therefore, by Definition 2.1.1, there exists a morphism $h:Q \to A$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "1_A"'] & Q \arrow[d, "g_i"] \arrow[dl, dashed, "h"'] \\ A \arrow[r, "f_i"'] & B_i \end{tikzcd}$$

commutes for each $i\varepsilon I$. Hence $h \cdot e = 1_A$, so that e is both a section and an epimorphism. Thus e is an isomorphism.

<u>(b) $\Rightarrow$ (c)</u>: Clear from Definition 2.1.1.

PROPOSITION 2.1.3. Let I and $\mathcal{C}$ be categories and let $D: I \to \mathcal{C}$ be a functor. If (L, ℓ_i) is a limit of D, then it is a strong mono source.

PROOF: By the definition of limit (1.1.6), (L, ℓ_i) is a natural source for D; i.e., for each I-morphism $m: i \to j$, $D(m) \cdot \ell_i = \ell_j$. Let X be a $\mathcal{C}$-object and let f and $g: X \to L$ be $\mathcal{C}$-morphisms such that $\ell_i \cdot f = \ell_i \cdot g$ for each $i \in Ob(I)$. Then $D(m) \cdot (\ell_i \cdot f) = (D(m) \cdot \ell_i) \cdot f = \ell_j \cdot f$ for each I-morphism $m: i \to j$. Hence $(X, (\ell_i \cdot f)_{i \in Ob(I)})$ is a natural source for D. But (L, ℓ_i) is a limit for D, thus there exists a <u>unique</u> morphism $h: X \to L$, such that $\ell_i \cdot f = \ell_i \cdot h$ for all $i \in Ob(I)$. Therefore $f = h = g$, and consequently (L, ℓ_i) is a mono source (2.1.1).

Suppose that (B, g_i) is a source and k and e are $\mathcal{C}$-morphisms where e is an epimorphism such that for each $i \in Ob(I)$ $g_i \cdot e = \ell_i \cdot k$. Then for each I-morphism $m: i \to j$, $D(m) \cdot g_i \cdot e = D(m) \cdot \ell_i \cdot k = \ell_j \cdot k = g_j \cdot e$, since (L, ℓ_i) is a natural source for D. However since e is an epimorphism, $D(m) \cdot g_i = g_j$. Thus (B, g_i) is a natural source for D. By Definition 2.1.1, there exists a unique morphism $k: B \to L$ such that $\ell_i \cdot k = g_i$ for each $i \in Ob(I)$. Hence, since (L, ℓ_i) is a mono source, the diagram

$$\begin{tikzcd} {} \arrow[r, "e"] \arrow[d, "k"'] & B \arrow[dl, dashed, "k"'] \arrow[d, "g_i"] \\ L \arrow[r, "\ell_i"'] & D(i) \end{tikzcd}$$

commutes for each $i \in Ob(I)$. Thus (L, ℓ_i) is a strong mono source (2.1.1).

It is well known that products, equalizers, terminal objects and pullbacks are special types of limits; hence they are examples of strong mono sources, and consequently they are examples of extremal mono sources. The following corollary provided the motivation for Proposition 2.1.3.

COROLLARY 2.1.4 (Herrlich and Strecker [7]). Let I and $\mathcal{C}$ be categories and let $D: I \to \mathcal{C}$ be a functor. If (L, ℓ_i) is a limit of D, then it is an extremal mono source.

PROOF: Propositions 2.1.3 and 2.1.2.

PROPOSITION 2.1.5. Let $\mathcal{C}$ be a category with pushouts. A source (A, f_i) in $\mathcal{C}$ is an extremal mono source in $\mathcal{C}$ if and only if it is a strong mono source in $\mathcal{C}$.

PROOF: By Proposition 2.1.3, we need only show that if (A, f_i) is an extremal mono source, then it is a strong mono source. Suppose that e is an epimorphism and for each $i \in I$, the diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "k"'] & B \arrow[d, "g_i"] \\ A \arrow[r, "f_i"'] & C_i \end{tikzcd}$$

commutes. Let

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "k"'] & B \arrow[d, "p"] \\ A \arrow[r, "q"'] & P \end{tikzcd}$$

be the pushout of e and k. Then for each $i\in I$, there exists a morphism $h_i:P \to C_i$ such that $f_i = h_i\cdot q$. But since pushouts preserve epimorphisms, q is an epimorphism; hence q is an isomorphism, since (A,f_i) is an extremal mono source.

Then $q^{-1}\cdot p:B \to A$ and for each $i\in I$, $f_i\cdot q^{-1}\cdot p\cdot e = h_i\cdot q\cdot q^{-1}\cdot e = g_i\cdot e = f_i\cdot k$ so that since e is an epimorphism and (A,f_i) is a mono source, the diagram

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "k"'] & B \arrow[d, "g_i"] \arrow[dl, "q^{-1}\cdot p"'] \\
A \arrow[r, "f_i"'] & C_i
\end{tikzcd}$$

commutes for each $i\in I$.

The following proposition was proved for extremal sources by Herrlich and Strecker [7].

PROPOSITION 2.1.6. Let $\mathcal{C}$ be a category with coequalizers. Then each extremal source (resp., strong source) in $\mathcal{C}$ is an extremal mono source (resp., strong mono source) in $\mathcal{C}$.

PROOF: Let (A,f_i) be an extremal source (resp., strong source) in $\mathcal{C}$, and let p and q be $\mathcal{C}$-morphisms such that $f_i\cdot p = f_i\cdot q$ for each $i\in I$. Let (k,K) be a coequalizer for p and q. Then k is an epimorphism, and for every f_i, such that $f_i\cdot p = f_i\cdot q$, there exists a unique morphism $g_i:K \to C_i$ such that the diagram

$$\begin{tikzcd} & & K \arrow[d, dashed, "g_i"] \\ X \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & A \arrow[ur, "k"] \arrow[r, "f_i"'] & C_i \end{tikzcd}$$

commutes. Since (A,f_i) is an extremal source (resp., a strong source, hence an extremal source by Proposition 2.1.2), k must be an isomorphism (2.1.1). Hence $p = q$; so (A,f_i) is a mono source.

PROPOSITION 2.1.7. Let $\mathcal{C}$ be a finitely cocomplete category and let (A,f_i) be a source in $\mathcal{C}$. The following statements are equivalent.

(a) (A,f_i) is a strong source

(b) (A,f_i) is a strong mono source

(c) (A,f_i) is an extremal source

(d) (A,f_i) is an extremal mono source.

PROOF: Since $\mathcal{C}$ is finitely cocomplete, it has coequalizers and pushouts.Thus (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) (2.1.6) and (d) $\Rightarrow$ (b) (2.1.5). Also it is clear that (b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) (2.1.1) and (b) $\Rightarrow$ (d) (2.1.2).

§2.2 Sources in Concrete Categories

DEFINITION 2.2.1. Let $(\mathcal{C},\mathcal{U})$ be a concrete category.

(1) A source (A,f_i) is called a <u>concrete embedding source</u> provided that it is a mono source and for every source (B,g_i) for which there exists a function $h\colon \mathcal{U}(B) \to \mathcal{U}(A)$ such that the diagram

$$\begin{tikzcd}
\mathcal{U}(B) \arrow[d, "h"'] \arrow[dr, "\mathcal{U}(g_i)"] & \\
\mathcal{U}(A) \arrow[r, "\mathcal{U}(f_i)"'] & \mathcal{U}(C_i)
\end{tikzcd}$$

commutes for each $i \varepsilon I$, there exists a morphism $\overline{h}:B \to A$ such that $\mathcal{U}(\overline{h}) = h$.

(2) A source (A,f_i) is called a <u>strong concrete embedding source</u> (respectively, an <u>extremal concrete embedding source</u>) provided that it is both a concrete embedding source and a strong (resp., extremal) source.

The properties of concrete embedding sources will be examined in subsequent sections within the generalized framework of $\mathcal{M}$-sources.

<u>§2.3 $\mathcal{M}$-Sources</u>

Now that we have defined several distinct types of sources, we will find it convenient to introduce some unifying notation.

DEFINITION 2.3.1. Let $\mathcal{C}$ be a category and let $\mathcal{M}$ be any class of morphisms in $\mathcal{C}$.

(1) A $\mathcal{C}$ -morphism f will be called an $\mathcal{M}$ -<u>morphism</u> in $\mathcal{C}$ provided that $f \varepsilon \mathcal{M}$.

(2) $\mathcal{M}$ will be called <u>isomorphism-closed in</u> $\mathcal{C}$ provided that for any $\mathcal{M}$ -morphism m and any $\mathcal{C}$-isomorphisms e and e', such that the compositions $m \cdot e$ and $e' \cdot m$ are defined in $\mathcal{C}$, $m \cdot e$ and $e' \cdot m$ must be $\mathcal{M}$-morphisms, and $\mathcal{M}$ must contain all of the isomorphisms in $\mathcal{C}$.

(3) $\mathcal{M}$ will be called <u>left-cancellative in</u> $\mathcal{C}$ provided that

whenever $p \cdot q \in \mathcal{M}$ for $\mathcal{C}$-morphisms p and q, q must be an $\mathcal{M}$-morphism.

(4) A pair (X,f) will be called an $\mathcal{M}$-subobject of Y provided that X and Y are $\mathcal{C}$-objects and $f: X \to Y$ is an $\mathcal{M}$-morphism.

For certain classes $\mathcal{M}$ of morphisms in $\mathcal{C}$, we have previously defined sources whose morphisms in the aggregate exhibit properties similar to the properties of individual $\mathcal{M}$-morphisms. These sources, as listed below in Table 2.3.2, will be called "$\mathcal{M}$-sources." We will refer to Table 2.3.2 frequently throughout the remainder of the paper. Note that each class of morphisms listed in Table 2.3.2 is isomorphism-closed in $\mathcal{C}$.

TABLE 2.3.2. Let $\mathcal{C}$ be a category. [Let $(\mathcal{C}, \mathcal{U})$ be a concrete category.]

$\mathcal{M}$	$\mathcal{M}$-morphism	$\mathcal{M}$-subobject	$\mathcal{M}$-source
the class of all morphisms in $\mathcal{C}$	morphism	weak subobject	source
the class of all monomorphisms in $\mathcal{C}$	monomorphism	subobject	mono source
the class of all extremal morphisms in $\mathcal{C}$	extremal morphism	extremal weak subobject	extremal source
the class of all extremal monomorphisms in $\mathcal{C}$	extremal monomorphism	extremal subobject	extremal mono source
the class of all strong morphisms in $\mathcal{C}$	strong morphism		strong source
the class of all strong monomorphisms in $\mathcal{C}$	strong monomorphism	strong subobject	strong mono source

TABLE 2.3.2 (Continued)

$\mathcal{M}$	$\mathcal{M}$-morphism	$\mathcal{M}$-subobject	$\mathcal{M}$-source
the class of all concrete embeddings in $\mathcal{C}$	concrete embedding	concrete embedded subobject	concrete embedding source
the class of all extremal concrete embeddings in $\mathcal{C}$	extremal concrete embedding	extremal concrete embedded subobject	extremal concrete embedding source
the class of all strong concrete embeddings in $\mathcal{C}$	strong concrete embedding	strong concrete embedded subobject	strong concrete embedding source

PROPOSITION 2.3.3 (Singleton $\mathcal{M}$-Sources). Let $\mathcal{C}$ be a category. [Let $(\mathcal{C},\mathcal{U})$ be a concrete category.] Let $\mathcal{M}$ be a class of morphisms in $\mathcal{C}$ listed in Table 2.3.2, and let $f:A \to B$ be a morphism in $\mathcal{C}$. Then the following statements are equivalent.

(a) (A,f) is an $\mathcal{M}$-source

(b) $f:A \to B$ is an $\mathcal{M}$-morphism

(c) (A,f) is an $\mathcal{M}$-subobject of B

PROOF: (b) $\leftrightarrow$ (c): Apply Definition 2.3.1.

(a) $\leftrightarrow$ (b): Apply Definitions 2.1.1 and 1.3.1 when $\mathcal{C}$ is any category and Definitions 2.2.1 and 1.5.1 when $(\mathcal{C},\mathcal{U})$ is a concrete category. For example: (A,f) is a concrete embedding source provided that for every source (B,g), for which there exists a function $h: \mathcal{U}(B) \to \mathcal{U}(A)$ such that $\mathcal{U}(f)\cdot h = \mathcal{U}(g)$, there must exist a morphism $\overline{h}:A \to B$ such that $\mathcal{U}(\overline{h}) = h$ (2.2.1). This condition holds if and only if f is a concrete embedding (1.5.1).

Thus, from the above result we can see that several of the propositions on $\mathcal{M}$-sources in §2.1 will automatically yield results on $\mathcal{M}$-morphisms; e.g., every strong morphism is an extremal morphism (2.1.2); in categories with pushouts, every extremal monomorphism is a strong monomorphism (2.1.5); and in categories with coequalizers, every strong (resp., extremal) morphism is a strong (resp., extremal) monomorphism (2.1.6).

PROPOSITION 2.3.4 (Enlargement of $\mathcal{M}$-Sources). Let $\mathcal{C}$ be a category. [Let $(\mathcal{C},\mathcal{U})$ be a concrete category.] Let $\mathcal{M}$ be a class of morphisms in $\mathcal{C}$ listed in Table 2.3.2, (A,f_i) be a source and $(f_k)_{k\varepsilon K}$ be a family of morphisms in $\mathcal{C}$ with domain A, having $(f_i)_{i\varepsilon I}$ as a subfamily. Then if $(A,f_i)_I$ is an $\mathcal{M}$-source, so is $(A,f_k)_K$.

PROOF: Clearly (A,f_k) is a source.

(a) <u>mono</u>: Let p and q be $\mathcal{C}$-morphisms such that $f_k \cdot p = f_k \cdot q$ for all $k\varepsilon K$. Then $f_i \cdot p = f_i \cdot q$ for all $i\varepsilon I$, so that since (A,f_i) is a mono source, $p = q$. Consequently (A,f_k) is a mono source.

(b) <u>extremal</u>: Suppose $(g_k)_{k\varepsilon K}$ is a family of $\mathcal{C}$-morphisms and e is an epimorphism in $\mathcal{C}$ such that $f_k = g_k \cdot e$ for all $k\varepsilon K$. Then $f_i = g_i \cdot e$ for all $i\varepsilon I$, so that since (A,f_i) is an extremal source, e is an isomorphism (2.1.1). Hence (A,f_k) is an extremal source.

(c) <u>strong</u>: Suppose (B,g_k) is a source, e is an epimorphism and r is a morphism such that $g_k \cdot e = f_k \cdot r$ for all $k\varepsilon K$. Then $g_i \cdot e = f_i \cdot r$ for all $i\varepsilon I$. Thus, since (A,f_i) is a strong source, there exists $h:B \to A$ such that the diagram

$$\begin{tikzcd} {} \arrow[r, "e"] \arrow[d, "r"'] & B \arrow[d, "g_i"] \arrow[dl, dashed, "h"'] \\ A \arrow[r, "f_i"'] & C_i \end{tikzcd}$$

commutes for all $i \in I$. Hence $r = h \cdot e$. Now $f_k \cdot h \cdot e = g_k \cdot e$ for all $k \in K$. But e is an epimorphism, hence $f_k \cdot h = g_k$ for all $k \in K$. Thus (A, f_k) is a strong source.

(d) <u>concrete embedding</u>: Suppose (B, g_k) is a source for which there exists a function $h\colon \mathcal{U}(B) \to \mathcal{U}(A)$ such that $\mathcal{U}(f_k) \cdot h = \mathcal{U}(g_k)$ for all $k \in K$. Then $\mathcal{U}(f_i) \cdot h = \mathcal{U}(g_i)$ for all $i \in I$. Hence, since (A, f_i) is a concrete embedding source, there exists $\overline{h}\colon B \to A$ such that $\mathcal{U}(\overline{h}) = h$ (2.2.1), and consequently (A, f_k) is a concrete embedding source.

(e) The remainder of the proof follows directly from parts (a), (b), (c), and (d).

PROPOSITION 2.3.5 (Left-Cancellation of $\mathcal{M}$-Sources). Let $\mathcal{C}$ be a category. [Let $(\mathcal{C}, \mathcal{U})$ be a concrete category.] And let $\mathcal{M}$ be any class of morphisms in $\mathcal{C}$ listed in Table 2.3.2. Let (A, f_i) be a source in $\mathcal{C}$ for which there exist families $(h_i)_{i \in I}$ and $(g_i)_{i \in I}$ of morphisms in $\mathcal{C}$ such that $f_i = h_i \cdot g_i$ for all $i \in I$. Then if (A, f_i) is an $\mathcal{M}$-source, so is (A, g_i).

PROOF: Clearly (A, g_i) is a source in $\mathcal{C}$.

(a) <u>mono</u>: Let p and q be $\mathcal{C}$-morphism such that $g_i \cdot p = g_i \cdot q$. Then $f_i \cdot p = h_i \cdot g_i \cdot p = h_i \cdot g_i \cdot q = f_i \cdot q$. Since (A, f_i) is a mono source, $p = q$; thus (A, g_i) a mono source (2.1.1).

(b) <u>extremal</u>: Let $(r_i)_{i\varepsilon I}$ be a family of $\mathcal{C}$-morphisms and let e be an epimorphism such that $g_i = r_i \cdot e$ for all $i\varepsilon I$. Then $f_i = h_i \cdot r_i \cdot e$ for all $i\varepsilon I$. Since (A,f_i) is an extremal source, e is an isomorphism; thus (A,g_i) is an extremal source (2.1.1).

(c) <u>strong</u>: Suppose (B,r_i) is a source in $\mathcal{C}$, e is an epimorphism in $\mathcal{C}$ and k is a morphism in $\mathcal{C}$ such that $r_i \cdot e = g_i \cdot k$ for all $i\varepsilon I$. Then $h_i \cdot r_i \cdot e = h_i \cdot g_i \cdot k = f_i \cdot k$. Since (A,f_i) is a strong source, there exists $p: B \to A$ such that the diagram

$$\begin{tikzcd}
\cdot \arrow[rr, "e"] \arrow[dd, "k"'] & & B \arrow[d, "r_i"] \arrow[ddll, dashed, "p"'] \\
& & G_i \arrow[d, "h_i"] \\
A \arrow[r, "g_i"] \arrow[rr, bend right, "f_i"'] & G_i \arrow[r, "h_i"] & C_i
\end{tikzcd}$$

commutes for all $i\varepsilon I$ (2.1.1); hence $p \cdot e = k$ and $r_i \cdot e = g_i \cdot (p \cdot e)$. However, e is an epimorphism, so $r_i = g_i \cdot p$ for all $i\varepsilon I$. Thus (A,g_i) is a strong source.

(d) <u>concrete embedding</u>: Suppose (B,r_i) is a source in $\mathcal{C}$ for which there exists a function $h: \mathcal{U}(B) \to \mathcal{U}(A)$ such that $\mathcal{U}(g_i) \cdot h = \mathcal{U}(r_i)$ for all $i\varepsilon I$. Since $f_i = h_i \cdot g_i$ for all $i\varepsilon I$ and functors preserve composition, $\mathcal{U}(f_i) \cdot h = \mathcal{U}(h_i \cdot g_i) \cdot h = \mathcal{U}(h_i) \cdot \mathcal{U}(g_i) \cdot h = \mathcal{U}(h_i) \cdot \mathcal{U}(r_i) = \mathcal{U}(h_i \cdot r_i)$ for all $i\varepsilon I$. And because (A,f_i) is a concrete embedding source, there exists a morphism $\overline{h}: B \to A$ such that $\mathcal{U}(\overline{h}) = h$ (2.2.1) and, consequently, (A,g_i) is a concrete embedding source.

(e) The remainder of the proof follows directly from parts (a), (b), (c) and (d).

COROLLARY 2.3.6. Let $\mathcal{M}$ be any class of $\mathcal{C}$-morphisms listed in Table 2.3.2. Then $\mathcal{M}$ is left-cancellative in $\mathcal{C}$.

PROOF: Suppose p and q are morphisms in $\mathcal{C}$, such that $p \cdot q$ is an $\mathcal{M}$-morphism. Then $(A, p \cdot q)$ is an $\mathcal{M}$-source (2.3.3) and consequently (A, q) is an $\mathcal{M}$-source (2.3.5). Thus q is an $\mathcal{M}$-morphism (2.3.3).

PROPOSITION 2.3.7. Let $\mathcal{C}$ be a category [$(\mathcal{C}, \mathcal{U})$ be a concrete category] and let $\mathcal{M}$ be any class of $\mathcal{C}$-morphisms listed in Table 2.3.2. Then $\mathcal{M}$ contains all of the sections in $\mathcal{C}$.

PROOF: (a) <u>mono</u>: Clear.

(b) <u>strong</u>: We will show that every section is a strong monomorphism. Let s be a section. Suppose that k and g are morphisms and e is an epimorphism such that $s \cdot k = g \cdot e$. Now there exists a morphism $\hat{s}$ such that $\hat{s} \cdot s = 1$. Thus $s \cdot \hat{s} \cdot g \cdot e = s \cdot \hat{s} \cdot s \cdot k = s \cdot k = g \cdot e$, so that since e is an epimorphism and s is a monomorphism, the diagram

$$\begin{tikzcd}
{} \arrow[r, "e"] \arrow[d, "k"'] & {} \arrow[d, "g"] \arrow[dl, dashed, "\hat{s} \cdot g"] \\
{} \arrow[r, "s"'] & {}
\end{tikzcd}$$

commutes.

(c) <u>extremal</u>: Every section is a strong monomorphism, hence an extremal monomorphism (2.1.2). Thus every section is extremal.

(d) <u>concrete embedding</u>: We will show that each section is a concrete embedding. Suppose that $s:A \to B$ is a section, $g:C \to B$ is a morphism and $h: \mathcal{U}(C) \to \mathcal{U}(A)$ is a function such that $\mathcal{U}(s)\cdot h = \mathcal{U}(g)$. Now there is a morphism $\hat{s}$ such that $\hat{s}\cdot s = 1_A$. Thus $h = \mathcal{U}(\hat{s})\cdot \mathcal{U}(s)\cdot h = \mathcal{U}(\hat{s})\cdot \mathcal{U}(g) = \mathcal{U}(\hat{s}\cdot g)$. Thus s is a concrete embedding.

The remainder of the proof follows directly from parts (a), (b), (c) and (d).

3. EMBEDDINGS INTO PRODUCTS

In this chapter we shall investigate the characteristics of categorical 'embeddings into products,' and prove an embedding theorem, which interrelates $\mathcal{M}$-sources and the existence of $\mathcal{M}$-morphisms from the object part of the source into the product of codomains of the source morphisms. We shall examine the concept of 'embeddable objects,' arriving at generalized definitions for $\mathcal{A}$-regular and $\mathcal{A}$-compact objects. These definitions will enable us to prove characterization theorems for epireflective subcategories and epireflective hulls in certain concrete categories.

§3.1 Embedding Theorems

Herrlich and Strecker [7] proved the portions of the following embedding theorem that deal with mono sources and extremal mono sources. Their work provided the motivation for the following generalized result.

THEOREM 3.1.1 (The Embedding Theorem). Let $\mathcal{C}$ be a category. [Let $(\mathcal{C}, \mathcal{U})$ be a concrete category such that $\mathcal{U}$ preserves monomorphisms and products.] Let $\mathcal{M}$ be a class of $\mathcal{C}$-morphisms listed in Table 2.3.2, and let (X, f_i) be a set-indexed source, with each f_i having codomain E_i, such that a product $(\Pi E_i, \pi_i)$ of the codomains exists in $\mathcal{C}$. Then the unique induced morphism $\langle f_i \rangle : X \to \Pi E_i$ is an $\mathcal{M}$-morphism if and only if (X, f_i) is an $\mathcal{M}$-source.

PROOF: (a) <u>mono</u>: Suppose that p and q are any $\mathcal{C}$-morphisms such that $f_i \cdot p = f_i \cdot q$ for all $i \in I$. Then $\pi_i \cdot \langle f_i \rangle \cdot p = \pi_i \cdot \langle f_i \rangle \cdot q$ for each $i \in I$.

But $(\Pi E_i, \pi_i)$ is a mono source (2.1.3 and 2.1.2), thus $\langle f_i \rangle \cdot p = \langle f_i \rangle \cdot q$. Consequently, (X, f_i) is a mono source if and only if $\langle f_i \rangle$ is a monomorphism (2.1.1).

(b) <u>extremal</u>: Suppose that $\langle f_i \rangle$ is an extremal morphism and that (B, g_i) is a source in $\mathcal{C}$ for which there is an epimorphism e in $\mathcal{C}$ such that $f_i = g_i \cdot e$ for each $i \varepsilon I$. By the definition of product (1.1.2), there exists a unique morphism $\langle g_i \rangle : B \to \Pi E_i$ such that $\pi_i \cdot \langle g_i \rangle = g_i$ for each $i \varepsilon I$. Then $\pi_i \cdot \langle f_i \rangle = f_i = g_i \cdot e = \pi_i \cdot \langle g_i \rangle \cdot e$. Since $(\Pi E_i, \pi_i)$ is a mono source (2.1.3), $\langle f_i \rangle = \langle g_i \rangle \cdot e$. And because $\langle f_i \rangle$ is an extremal morphism, e must be an isomorphism (1.3.1); consequently, (X, f_i) is an extremal source (2.1.1). Conversely, suppose that (X, f_i) is an extremal source and that $g \cdot e = \langle f_i \rangle$ is a factorization of $\langle f_i \rangle$ for which e is an epimorphism. Let Y denote the codomain of e. Then $(Y, \pi_i \cdot g)$ is a source and $f_i = \pi_i \cdot \langle f_i \rangle = (\pi_i \cdot g) \cdot e$ for each $i \varepsilon I$. Since (A, f_i) is an extremal source, e must be an isomorphism. Hence $\langle f_i \rangle$ must be an extremal morphism (1.3.1).

(c) <u>strong</u>: Suppose (B, g_i) is any source in $\mathcal{C}$ and e is any epimorphism in $\mathcal{C}$ and k is any morphism in $\mathcal{C}$ such that $f_i \cdot k = g_i \cdot e$ for each $i \varepsilon I$. Then there exists a unique morphism $\langle g_i \rangle : B \to \Pi E_i$ such that $\pi_i \cdot \langle g_i \rangle = g_i$ for each $i \varepsilon I$. If (X, f_i) is a strong source, there exists a morphism $h : B \to X$ such that the inner portion of the diagram

$$\begin{tikzcd}
{} \arrow[r, "e"] \arrow[d, "k"'] & B \arrow[dl, dashed, "h"] \arrow[d, "g_i"] \arrow[dr, bend left, "\langle g_i \rangle"] & \\
X \arrow[r, "f_i"'] \arrow[rr, bend right, "\langle f_i \rangle"'] & E_i & \Pi E_i \arrow[l, "\pi_i"']
\end{tikzcd}$$

commutes for all $i\epsilon I$; i.e., $k = h\cdot e$ and $f_i \cdot h = g_i$ for all $i\epsilon I$. Hence $\pi_i \cdot \langle f_i \rangle \cdot h = \pi_i \cdot \langle g_i \rangle$ for each $i\epsilon I$. Therefore, since $(\Pi E_i, \pi_i)$ is a mono source, $\langle f_i \rangle \cdot h = \langle g_i \rangle$. Hence the outer portion of the diagram above commutes, which implies that $\langle f_i \rangle$ is a strong morphism (1.3.1). Conversely, suppose that $\langle f_i \rangle$ is a strong morphism. Then there exists a morphism $h: B \to X$ such that the outer portion of the diagram commutes; i.e., $k = h\cdot e$ and $\langle f_i \rangle \cdot h = \langle g_i \rangle$. Clearly $f_i \cdot h = \pi_i \cdot \langle f_i \rangle \cdot h = \pi_i \cdot \langle g \rangle = g_i$ for each $i\epsilon I$, so that the inner portion of the diagram commutes for each $i\epsilon I$. Hence (X, f_i) is a strong source (2.1.1).

(d) <u>concrete embedding</u>: From part (a), (X, f_i) is a mono source if and only if $\langle f_i \rangle$ is a monomorphism. Suppose that (B, g_i) is any source in $\mathcal{C}$ for which there exists a function $h: \mathcal{U}(B) \to \mathcal{U}(X)$ such that $\mathcal{U}(f_i) \cdot h = \mathcal{U}(g_i)$ for each $i\epsilon I$. Then by the definition of product (1.1.2), there exists a unique morphism $\langle g_i \rangle : B \to \Pi E_i$ such that $\pi_i \cdot \langle g_i \rangle = g_i$ for all $i\epsilon I$. By hypothesis, $\mathcal{U}$ preserves products; hence $(\mathcal{U}(\Pi E_i), \mathcal{U}(\pi_i))$ is a product in <u>Set</u>, hence a mono source in <u>Set</u> (2.1.3). Thus $\mathcal{U}(\pi_i) \cdot \mathcal{U}(\langle f_i \rangle) \cdot h = \mathcal{U}(\pi_i \cdot \langle f_i \rangle) \cdot h = \mathcal{U}(f_i) \cdot h$ $= \mathcal{U}(g_i) = \mathcal{U}(\pi_i \cdot \langle g_i \rangle) = \mathcal{U}(\pi_i) \cdot \mathcal{U}(\langle g_i \rangle)$ for all $i\epsilon I$ if and only if $\mathcal{U}(\langle f_i \rangle) \cdot h = \mathcal{U}(\langle g_i \rangle)$. Consequently (X, f_i) is a concrete embedding source if and only if $\langle f_i \rangle$ is a concrete embedding (2.2.1 and 1.5.1).

(e) The remainder of the proof follows directly from parts (a), (b), (c), and (d).

There are many applications for this theorem since the category $\mathcal{C}$ is quite unrestricted. A few of the examples are listed as corollaries.

COROLLARY 3.1.2. Let A be any set with at least two elements.

Then for any set S, there exists an injection into some power of A.

PROOF: Let a and b be distinct elements of A, and let $\mathfrak{F}$ be the set of all functions from S to A. Then $(S, \mathfrak{F})$ is a mono source in <u>Set</u>. To see this, suppose that p and q are distinct functions from some set X into S. Then for some $x \varepsilon X$, $p(x) \neq q(x)$. Define $\overline{f}: S \to A$ by

$$\overline{f}(s) = \begin{cases} a \text{ if } s = p(x) \\ b \text{ if } s \neq p(x) \end{cases}.$$

Then $\overline{f} \cdot p \neq \overline{f} \cdot q$. Since <u>Set</u> has products, the power $(A^{\mathfrak{F}}, \pi_f)$ exists in <u>Set</u>. Let $\langle f_i \rangle : S \to A^{\mathfrak{F}}$ be the unique induced function. By Theorem 3.1.1, since $(S, \mathfrak{F})$ is a mono source, $\langle f_i \rangle$ is an injection.

COROLLARY 3.1.3. A continuous map f from a topological space X into a topological product ΠE_i of spaces $(E_i)_{i \varepsilon I}$ is a topological embedding if and only if $(X, \pi_i \cdot f)$ is an extremal mono source in <u>Top</u>, where $\pi_i : \Pi E_i \to E_i$ are the projection maps.

PROOF: f is a topological embedding if and only if f is an extremal monomorphism in <u>Top</u> ([9]). Since the topological product together with the projection mappings form a categorical product in <u>Top</u>, f is an extremal monomorphism in <u>Top</u> if and only if $(X, \pi_i \cdot f)$ is an extremal mono source in <u>Top</u> (3.1.1).

COROLLARY 3.1.4. A continuous map f from a Hausdorff space X into a topological product of Hausdorff spaces, ΠE_i, is a homeomorphism onto a closed subset of the product if and only if $(X, \pi_i \cdot f)$ is an extremal mono source in <u>Haus</u> (where $\pi_i : \Pi E_i \to E_i$ are the projection maps).

PROOF: Extremal monomorphisms in <u>Haus</u> are exactly the topological embeddings onto closed subsets ([9]). Since a topological product

together with the projection mappings is the categorical product in <u>Haus</u>, we have, by Theorem 3.1.1, that f is an extremal monomorphism if and only if $(X, \pi_i \cdot f)$ is an extremal mono source in <u>Haus</u>.

COROLLARY 3.1.5. Let $(\mathcal{C}, \mathcal{U})$ be a complete, well-powered concrete category for which $\mathcal{U}$ preserves monomorphisms and products. Let (A, f_i) be a set-indexed source in $\mathcal{C}$. Then (A, f_i) is an extremal mono source in $\mathcal{C}$ if and only if it is an extremal concrete embedding source in $\mathcal{C}$.

PROOF: $\mathcal{C}$ is a complete category. Hence $(A, A \xrightarrow{f_i} E_i)$ being a set-indexed source, implies that the product $(\Pi E_i, \pi_i)$ exists in $\mathcal{C}$. If (A, f_i) is an extremal mono source, the induced morphism $\langle f_i \rangle : A \to \Pi E_i$ is an extremal monomorphism (3.1.1) and, consequently, $\langle f_i \rangle$ is an extremal concrete embedding (1.5.3). Thus (A, f_i) is an extremal concrete embedding source. Conversely, by definition, every extremal concrete embedding source is an extremal mono source (2.2.1).

A study of extensions naturally goes hand in hand with the study of embeddings.

DEFINITION 3.1.6. Let $\mathcal{C}$ be a category, $\mathcal{M}$ be any isomorphism-closed class of $\mathcal{C}$-morphisms, $\mathcal{E}$ be a subcategory of $\mathcal{C}$, and X be a $\mathcal{C}$-object.

(1) The pair (γ, Y) is called an <u>$\mathcal{M}$-extension of X in $\mathcal{E}$</u> provided that Y is an $\mathcal{E}$-object, (X, γ) is an $\mathcal{M}$-subobject of Y and γ is an epimorphism.

(2) Let (X, f_i) be a source with codomains in $\mathcal{E}$ (i.e., for each $i \in I$, the codomain E_i of f_i is an $\mathcal{E}$-object). (X, f_i) is called an

$\mathcal{M}$-nonextendable source with respect to $\mathcal{E}$ provided that for any $\mathcal{M}$-extension (γ, Y) of X in $\mathcal{C}$ with the property--for each $i \in I$, there exists a morphism $f_i^*: Y \to E_i$ for which the diagram

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "f_i"'] & & Y \arrow[dl, dashed, "f_i^*"] \\ & E_i & \end{tikzcd}$$

commutes--γ must be an isomorphism (i.e., there must exist no proper $\mathcal{M}$-extension of X in $\mathcal{C}$ with this property).

In particular, if a source (X, f_i) contains all the morphisms from X to $\mathcal{E}$-objects, then (X, f_i) is an $\mathcal{M}$-nonextendable source with respect to $\mathcal{E}$ if and only if there exists no proper $\mathcal{M}$-extension (w, W) for X in $\mathcal{C}$ for which w is $\mathcal{E}$-extendable (1.4.2).

Note that every class of $\mathcal{M}$-morphisms listed in Table 2.3.2 is isomorphism-closed in $\mathcal{C}$. For convenience, when $\mathcal{M}$ is the class of all $\mathcal{C}$-morphisms, we will use the prefix 'weak' to replace '$\mathcal{M}$' in Definition 3.1.6. Hence we can use the term 'weak-extension' to denote an $\mathcal{M}$-extension when $\mathcal{M}$ is the class of all morphisms in $\mathcal{C}$.

The definition for a weak-nonextendable source (X, f_i) with respect to $\mathcal{E}$ (3.1.6) corresponds to Mrówka's [16] definition for the class $\{f_i : i \in I\}$ to be an '$\mathcal{E}$-nonextendable class for X.'

PROPOSITION 3.1.7. Let $\mathcal{C}$ be a category, $\mathcal{E}$ be a subcategory of $\mathcal{C}$, and (X, f_i) be a source with codomains in $\mathcal{E}$. Then (X, f_i) is a weak-nonextendable source with respect to $\mathcal{E}$ if and only if (X, f_i) is

an extremal source.

PROOF: Suppose (X,f_i) is a weak-nonextendable source with respect to $\mathcal{E}$. Suppose that (Y,g_i) is a source in $\mathcal{C}$ such that for some epimorphism $e:X \to Y$, the diagram

$$\begin{tikzcd} X \arrow[rr, "e"] \arrow[dr, "f_i"'] & & Y \arrow[dl, "g_i"] \\ & E_i & \end{tikzcd}$$

commutes for each $i \in I$. Then (e,Y) is a weak-extension; hence e must be an isomorphism (3.1.6). Consequently (X,f_i) is an extremal source (2.1.1).

Conversely, let (X,f_i) be an extremal source. Suppose there exists a weak-extension (γ,Y) such that, for each $i \in I$, there exists a morphism g_i for which $f_i = g_i \cdot \gamma$. But γ is an epimorphism (3.1.6) and (X,f_i) is an extremal source; hence γ must be an isomorphism (2.1.1). Thus (X,f_i) is a weak-nonextendable source with respect to $\mathcal{E}$.

Corollary 3.1.3 gives us a way to determine whether a continuous map from a topological space into a topological product of spaces is a topological embedding. And Corollary 3.1.4 gives us a way to determine whether a continuous map from a Hausdorff space into a product of Hausdorff spaces is a homeomorphism onto a closed subspace of the product. Yet these methods are not as convenient as we might wish. Mrówka's Embedding Theorem yields a topological characterization of these results. Parts of this theorem follow directly from Theorems 3.1.1 and 3.1.7.

THEOREM 3.1.8 (Mrówka's Embedding Theorem [16]). Let

$\mathfrak{F} = \{f_i : i \in I\}$ be a set of functions with $f_i : X \to E_i$ where X and E_i, for each $i \in I$, are topological spaces. Let h be the set function from X into the topological product ΠE_i such that $\pi_i \cdot h = f_i$ for each $i \in I$. We have

(a) h is continuous if and only if each f_i is continuous.

(b) h is injective if and only if the set $\mathfrak{F}$ satisfies the following condition:

C1. for every $p, q \in X$ such that $p \neq q$, there is an $f_i \in \mathfrak{F}$ with $f_i(p) \neq f_i(q)$.

(c) h is a topological embedding if and only if h is continuous and injective and the set $\mathfrak{F}$ satisfies the following condition:

C2. for every closed subset A of X and for every $p \in X \setminus A$, there exists a finite system $f_{i_1}, \ldots, f_{i_n}$ of functions from $\mathfrak{F}$ such that $\langle f_{i_1}, \ldots, f_{i_n} \rangle(p) \notin cl\langle f_{i_1}, \ldots, f_{i_n} \rangle[A]$ (where cl stands for closure in $\prod_{1 \leq k \leq n} E_{i_k}$).

(d) Assume that the spaces E_i are all Hausdorff and assume that h is a topological embedding. $h[X]$ is closed in ΠE_i if and only if the set $\mathfrak{F}$ satisfies the following condition:

C3. there is no proper weak extension (γ, Y) of X such that every function $f_i \in \mathfrak{F}$ admits a continuous extension such that the diagram

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "f_i"'] & & \gamma X \arrow[dl, dashed, "f_i^*"] \\ & E_i & \end{tikzcd}$$

commutes.

PROOF: Parts (a) and (b) are well-known results. Note that the topological product with the Tychonoff product topology is a product in Top. Hence each f_i is a Top-morphism, i.e., continuous, if and only if h is a Top-morphism. Also h is injective if and only if it is a Set-monomorphism, which by Theorem 3.1.1 is true if and only if condition C1 is fulfilled in Set. (Let p, $q \in X$ such that $p \neq q$ and let $g_p, g_q : \{y\} \to X$ be defined so that $g_p(y) = p$ and $g_q(y) = q$. Thus (X, f_i) is a mono source in Set if and only if C1 holds.)

(c) The reader is referred to Mrówka's proof [16].

(d) Assume that the spaces E_i are all Hausdorff and h is a topological embedding. Then h[X] is a Hausdorff space, so X is also a Hausdorff space and consequently, h is a concrete embedding. Let $\mathcal{E}$ be the full subcategory of Haus having $\{E_i : i \in I\}$ as its class of objects. Then C3 merely states that (X, f_i) is a weak-nonextendable source with respect to $\mathcal{E}$. Thus by Proposition 3.1.7, C3 holds if and only if (X, f_i) is an extremal source in Haus, which by Theorem 3.1.1 is true if and only if h is an extremal morphism in Haus; i.e., if and only if h is an extremal monomorphism in Haus (2.1.6); i.e., if and only if h[X] is closed.

COROLLARY 3.1.9. Under the same hypothesis as Theorem 3.1.8, the following statements hold:

(a) (X, f_i) is a mono source in Top if and only if (X, f_i) is a source in Top such that C1 holds.

(b) (X, f_i) is a concrete embedding source in Top if and only if (X, f_i) is a mono source in Top such that C2 holds.

PROOF: (a) (X, f_i) is a mono source in Top if and only if h is a monomorphism (3.1.1) which holds if and only if (X, f_i) is a source for which C1 holds (3.1.8).

(b) (X, f_i) is a concrete embedding source in Top if and only if h is a concrete embedding in Top (3.1.1) [but concrete embeddings are precisely the topological embeddings ([7])], which holds if and only if (X, f_i) is a source for which condition C2 holds (3.1.8).

§3.2 $\mathcal{E} \mid \mathcal{M}$-Embeddable Objects

Mrówka [15] defined an E-completely regular space to be a topological space that is homeomorphic to a subspace of some topological power E^m of the space E. For a Hausdorff space E, he defined an E-compact space to be a space that is homeomorphic to a closed subspace of some topological power E^m of E. For a given full subcategory $\mathcal{E}$ of the category Top (respectively, Haus), Herrlich [6] defined an $\mathcal{E}$-regular space (respectively, an $\mathcal{E}$-compact space) to be any object in the epireflective hull of $\mathcal{E}$ in Top (respectively, Haus). First we shall directly generalize Mrówka's definition in three ways: to arbitrary categories $\mathcal{C}$ (rather than Top or Haus); to arbitrary subcategories $\mathcal{E}$ of $\mathcal{C}$ (rather than subcategories with a single object E); and to arbitrary $\mathcal{M}$-morphisms, for various classes $\mathcal{M}$ of $\mathcal{C}$-morphisms (rather than topological embeddings) into the object parts of products of $\mathcal{E}$-objects. In later sections, §3.3 and §3.4, on epireflective subcategories, we will see that our definitions coincide with Herrlich's definitions in the categories Top and Haus.

DEFINITION 3.2.1. Let $\mathcal{C}$ be a category, $\mathcal{E}$ be a subcategory of $\mathcal{C}$,

and $\mathcal{M}$ be any class of $\mathcal{C}$-morphisms that is isomorphism-closed in $\mathcal{C}$.

(1) A $\mathcal{C}$-object X is called $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$ provided that there exists a set-indexed family $(E_i)_{i\in I}$ of $\mathcal{E}$-objects whose product $(\Pi E_i, \pi_i)$ exists in $\mathcal{C}$ and for which there exists an $\mathcal{M}$-morphism $f: X \to \Pi E_i$.

(2) When $\mathcal{E}$ has a single object E, the term $E|\mathcal{M}$-embeddable in $\mathcal{C}$ will be used interchangeably with $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$.

(3) Let $(\mathcal{C}, \mathcal{U})$ be a concrete category. Whenever $\mathcal{M}$ is the class of all concrete embeddings in $\mathcal{C}$, the term $\mathcal{E}$-regular in $\mathcal{C}$ will be used interchangeably with the term $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$. Whenever $\mathcal{M}$ is the class of all extremal concrete embeddings in $\mathcal{C}$, the term $\mathcal{E}$-compact in $\mathcal{C}$ will be used interchangeably with the term $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$.

Note that since the concrete embeddings in Top are precisely the topological embeddings, our definition for an E-regular space in Top coincides with Mrówka's definition for an E-completely regular space. Also our definition for a Hausdorff space to be E-compact in Haus coincides with Mrówka's definition of E-compactness, since the extremal concrete embeddings in Haus are homeomorphisms onto closed subspaces of Hausdorff spaces (1.5.3).

The study of topological spaces has provided the motivation for these definitions. Clearly, however, by Corollary 3.1.2, for any set A with at least two points, every set S is A-regular in Set. In order to illustrate the generality of these concepts, we will construct an algebraic example. Let AbMon be the category of all Abelian monoids and

monoid homomorphisms. Then Ab, the category of all Abelian groups and group homomorphisms, is a full subcategory of AbMon. The following proposition is a well-known result, although it may not be immediately recognizable in our new terminology.

PROPOSITION 3.2.2. An Abelian monoid is Ab-regular in AbMon if and only if it is cancellative.

PROOF: Suppose M is Ab-regular in AbMon, then there exists a family $(A_i)_{i\varepsilon I}$ of Abelian groups and a concrete embedding morphism $f:M \to \Pi A_i$. Suppose $a,b,m\varepsilon M$ and $m+a = m+b$. Then $f(m+a) = f(m+b)$ so that since f is a monoid homomorphism $f(m)*f(a) = f(m)*f(b)$. Thus $f(a) = f(b)$, since $(\Pi A_i,*)$ is a group. Hence a=b, since f is injective on the underlying sets (1.5.5). Consequently M is cancellative.

Conversely, let M be a cancellative monoid. We will define a relation R on pairs (x,y) where $x,y\varepsilon M$ in the following manner: $(x,y)R(x',y')$ iff $x+y' = y + x'$. It is straightforward to show that R is an equivalence relation. Let $\widetilde{R}$ be the set of all equivalence classes $\widetilde{(x,y)}$, and let * be the operation of componentwise addition of pairs, which is well-defined by the definition of R. Since M is cancellative and Abelian, it is easy to show that $(\widetilde{R},*)$ is an Abelian group, where $\widetilde{(0_M,0_M)} = \{(x,x):x\varepsilon M\}$ is the identity element and the inverse of any element $\widetilde{(x,y)}$ is the element $\widetilde{(y,x)}$. Define $h:M \to \widetilde{R}$ by $h(x) = \widetilde{(0_M,x)}$. Since $h(x+y) = \widetilde{(0_M,x+y)} = \widetilde{(0_M,x)} * \widetilde{(0_M,y)}$ $= h(x) * h(y)$, h is a homomorphism. Let $z\varepsilon Ker(h)$. Then $h(z)$ $= \widetilde{(0_M,z)} = \widetilde{(0_M,0_M)}$. Thus $(0_M,z)R(0_M,0_M)$. Hence $0_M + 0_M = z+0_M$; so $z = 0_M$. Consequently h is injective. Thus h is a monomorphism in Mon,

which is an algebraic category having AbMon as a full subcategory; hence h is a concrete embedding in Mon, and a concrete embedding in AbMon (1.5.6). Consequently M is Ab-regular.

THEOREM 3.2.3. Let $\mathcal{C}$ be a category with products. [Let $(\mathcal{C},\mathcal{U})$ be a concrete category with products such that $\mathcal{U}$ preserves monomorphisms and products.] Let $\mathcal{E}$ be a subcategory of $\mathcal{C}$, let $\mathcal{M}$ be a class of $\mathcal{C}$-morphisms from Table 2.3.2, and let X be a $\mathcal{C}$-object.

Then X is $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$ if and only if there exists a set $\mathcal{E}_X$ contained in $\mathrm{Ob}(\mathcal{E})$ such that $(X, \bigcup_{E\in\mathcal{E}_X} \hom_{\mathcal{C}}(X,E))$ is an $\mathcal{M}$-source in $\mathcal{C}$.

PROOF: Let X be $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$. Then there exists a set-indexed family $(E_i)_{i\in I}$ of $\mathcal{E}$-objects such that there exists an $\mathcal{M}$-morphism $f:X \to \Pi E_i$ (3.2.1). By the Embedding Theorem (3.1.1), $(X,\pi_i\cdot f)$ is an $\mathcal{M}$-source. Since an $\mathcal{M}$-source can be enlarged (2.3.4), $(X,\bigcup_{i\in I} \hom_{\mathcal{C}}(X,E_i))$ is an $\mathcal{M}$-source.

Conversely, let $\mathcal{E}_X$ be a set contained in $\mathrm{Ob}(\mathcal{E})$ such that $(X, \bigcup_{E\in\mathcal{E}_X} \hom_{\mathcal{C}}(X,E))$ is an $\mathcal{M}$-source. By hypothesis, $\mathcal{C}$ has products. Hence the product $(\prod_{E\in\mathcal{E}_X} E^{\hom_{\mathcal{C}}(X,E)}, \pi_g^E\cdot\pi_E)$ exists in $\mathcal{C}$ (by the Iteration of Products Theorem (1.1.3)). Let $\langle g\rangle : X \to \prod_{E\in\mathcal{E}_X} E^{\hom_{\mathcal{C}}(X,E)}$ be the unique induced morphism such that $\pi_g^E\cdot\pi_E\cdot\langle g\rangle = g$ for all $g\in\bigcup_{E\in\mathcal{E}_X} \hom_{\mathcal{C}}(X,E)$. Thus $\langle g\rangle$ is an

$$
\begin{tikzcd}
X \arrow[r, dashed, "\langle g\rangle"] \arrow[ddr, "g"'] & \prod_{E\in} E^{\hom_{\mathcal{C}}(X,E)} \arrow[d, "\pi_E"] \\
 & E^{\hom_{\mathcal{C}}(X,E)} \arrow[d, "\pi_g^E"] \\
 & E
\end{tikzcd}
$$

$\mathcal{M}$-morphism (3.1.1), so that X is $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$.

COROLLARY 3.2.4. Let $(\mathcal{C}, \mathcal{U})$ be a concrete category with products such that $\mathcal{U}$ preserves monomorphisms and products. Let $\mathcal{E}$ be a full subcategory of $\mathcal{C}$ and let X be a $\mathcal{C}$-object.

Then X is $\mathcal{E}$-regular (respectively, $\mathcal{E}$-compact) in $\mathcal{C}$ if and only if there exists a <u>set</u> $\mathcal{E}_X$ contained in $Ob(\mathcal{E})$ such that $(X, \bigcup_{E \in \mathcal{E}_X} \hom_{\mathcal{C}}(X,E))$ is a concrete (respectively, extremal concrete) embedding source in $\mathcal{C}$.

COROLLARY 3.2.5 (Mrówka [16]). A topological space X is E-regular if and only if the following two conditions are satisfied:

C1': for every $p,q \in X$, $p \neq q$, there is a continuous function $f:X \to E$ with $f(p) \neq f(q)$.

C2': for every closed subset A of X and every $p \in X \setminus A$, there is a finite number n and a continuous function $f':X \to E^n$ such that $f'(p) \notin \overline{f'[A]}$.

PROOF: Let $\mathcal{J} = \hom_{\mathcal{C}}(X,E)$. By Corollary 3.1.9, $(X, \mathcal{J})$ is a concrete embedding source if and only if conditions C1' and C2' (which are precisely C1 and C2 of Theorem 3.1.8 stated for $E_i = E$ for all $i \in I$) hold. By the Embedding Theorem (3.1.1), the induced morphism $\langle f \rangle : X \to E^{\mathcal{J}}$ is a concrete embedding morphism (hence X is E-regular (3.2.1)) if and only if $(X, \mathcal{J})$ is a concrete embedding source.

There are many well-known corollaries to the above theorem. Topological examples have been collected by Mrowka [13] and [16] and Herrlich [4]. Many of these corollaries had been established as separate theorems long before the development of a unifying theory.

Several of these well-known results are listed below as examples and are stated without proof.

EXAMPLE 3.2.6 (Tychonoff). A topological space is a completely regular T_1-space if and only if it is [0,1]-regular in Top. A Hausdorff space is compact if and only if it is [0,1]-compact in Haus.

EXAMPLE 3.2.7 (Mrowka [14]). A topological space is a completely regular T_1-space if and only if it is (0,1)-regular in Top. A Hausdorff space is realcompact if and only if it is (0,1)-compact in Haus.

EXAMPLE 3.2.8 (Alexandroff). Let F be the T_o-space with two points {a,b} in which the only proper closed set is {a}. A topological space is a T_o-space if and only if it is F-regular in Top.

EXAMPLE 3.2.9. Let W be the two-point indiscrete space. A topological space is indiscrete if and only if it is W-regular in Top.

Recall that a topological space is said to be zero dimensional provided that it has a base of closed-and-open sets.

EXAMPLE 3.2.10 (Alexandroff). Let D be the discrete space with two points. A topological space is a zero-dimensional T_o-space if and only if it is D-regular in Top. A zero-dimensional Hausdorff space is compact if and only if it is D-compact in Haus.

EXAMPLE 3.2.11 (Mrowka [13]). Let V be a topological space with three points {a,b,c} such that {a} is the only proper non-empty open subset. Every topological space is V-regular in Top.

Let us now consider another result of Mrowka's. Let L_m denote a space with m-elements and the finite complement topology. Let $\underline{Top}_1$ denote the category of T_1-spaces and continuous maps. By Proposition 1.5.8, any $\underline{Top}_1$-morphism $f:A \to B$ is a concrete embedding in $\underline{Top}_1$ if and only if it is a concrete embedding in Top.

EXAMPLE 3.2.12 (Mrówka [13]). There exists no T_1-space X such that every T_1-space is X-regular in $\underline{\text{Top}}_1$. However $\mathcal{L} = \{L_m : m \text{ is a cardinal}\}$ is a class of T_1-spaces such that every T_1-space of cardinality m can be topologically embedded into the topological power $(L_m)^m$.

Note that there exists no T_1-space X, such that every T_1-space is X-regular in $\underline{\text{Top}}_1$, or X-regular in Top, although by Example 3.2.11, every topological space is V-regular in Top. Clearly, V is not a T_1-space. Also we have seen that every T_1-space is $\mathcal{L}$-regular in $\underline{\text{Top}}_1$. This example illustrates the necessity for our generalized definition. Trivially, of course, in a category $\mathcal{C}$, every $\mathcal{C}$-object is $\mathcal{C}|\mathcal{M}$-embeddable in $\mathcal{C}$ if the class $\mathcal{M}$ of morphisms in $\mathcal{C}$ contains the identity morphisms.

The remainder of this section will be used to reformulate a topological result of Mrówka [16] into category-theoretic terms. Recall that for a category $\mathcal{C}$ and $\mathcal{C}$-object E, the contravariant hom functor of $\mathcal{C}$ with respect to E is the functor $\hom_{\mathcal{C}}(-,E): \mathcal{C}^{*} \to \underline{\text{Set}}$ defined so that $\hom_{\mathcal{C}}(-,E)(X) = \hom_{\mathcal{C}}(X,E)$ for every $\mathcal{C}$-object X, and $\hom_{\mathcal{C}}(-,E)(\phi) = \_\cdot\phi$ for each $\mathcal{C}$-morphism ϕ.

DEFINITION 3.2.13. Let $\mathcal{C}$ be a category and let $\mathcal{M}$ be an isomorphism-closed class of $\mathcal{C}$-morphisms. Let E and X be $\mathcal{C}$-objects. Then a pair (X^*, ϕ_X) is called an <u>$E|\mathcal{M}$-transformation of</u> X provided that the following conditions are satisfied:

K1: X^* is an $E|\mathcal{M}$-embeddable $\mathcal{C}$-object.

K2: $\phi_X : X \to X^*$ is an epimorphism in $\mathcal{C}$ such that $\hom_{\mathcal{C}}(-,E)\phi_X : \hom_{\mathcal{C}}(X^*,E) \to \hom_{\mathcal{C}}(X,E)$ is bijective.

THEOREM 3.2.14 (The Identification Theorem). Let $\mathcal{C}$ be a category with products [Let $(\mathcal{C}, \mathcal{U})$ be a concrete category with products where $\mathcal{U}$ preserves monomorphisms and products.], let $\mathcal{M}$ be a class of $\mathcal{C}$-morphisms from Table 2.3.2 and let $\mathcal{C}$ have the (epi, $\mathcal{M}$) factorization property. Let E be a $\mathcal{C}$-object.

Then for any $\mathcal{C}$-object X there exists an $E|\mathcal{M}$-transformation (X^*, ϕ_X) of X.

PROOF: Let X be a $\mathcal{C}$-object. By hypothesis, the product $(E^{\hom_{\mathcal{C}}(X,E)}, \pi_f)$ exists in $\mathcal{C}$. Let the unique induced morphism be $\langle f\rangle : X \to E^{\hom_{\mathcal{C}}(X,E)}$. By hypothesis, $\langle f\rangle = p\cdot\phi_X$ where p is an $\mathcal{M}$-morphism and ϕ_X is an epimorphism (1.3.3). Let X^* denote the codomain of ϕ_X. Then X^* is clearly $E|\mathcal{M}$-embeddable in $\mathcal{C}$ (3.2.1) and $\hom_{\mathcal{C}}(-,E)(\phi_X)$ is injective since ϕ_X is an epimorphism. For every $f \varepsilon \hom_{\mathcal{C}}(X,E)$, the diagram

$$\begin{tikzcd}
& X^* \arrow[dr, "p"] & \\
X \arrow[ur, "\phi_X"] \arrow[rr, dashed, "\langle f\rangle"] \arrow[dr, "f"'] & & E^{\hom_{\mathcal{C}}(X,E)} \arrow[dl, "\pi_f"] \\
& E &
\end{tikzcd}$$

commutes; hence $\pi_f\cdot P \;\varepsilon\; \hom_{\mathcal{C}}(X^*,E)$ and $f = \pi_f\cdot p\cdot\phi_X = \hom_{\mathcal{C}}(-,E)(\phi_X)(\pi_f\cdot p)$. Therefore $\hom_{\mathcal{C}}(-,E)(\phi_X)$ is surjective.

COROLLARY 3.2.15 (Mrówka [16]). For all topological spaces X and E, there exists an E-regular space X^* and a continuous surjective map $\phi : X \to X^*$ such that $\hom_{\underline{Top}}(-,E)(\phi) : \hom_{\underline{Top}}(X^*,E) \to \hom_{\underline{Top}}(X,E)$ is bijective.

PROOF: Top has the unique (epi, extremal concrete embedding) factorization property (1.3.5).

§3.3 $\mathcal{M}$-Coseparating Classes

In this section we will show that in a category with products, an object E in the category, with the property that every other object is $E|$mono-embeddable, must be a coseparator for the category. Recall that a $\mathcal{C}$-object C is called a coseparator for $\mathcal{C}$ provided that for any two distinct $\mathcal{C}$-morphisms p and $q:A \to B$, there exists a $\mathcal{C}$-morphism $x:B \to C$ such that $x\cdot p \neq x\cdot q$. We shall first prove a proposition relating coseparators and mono sources, which we will use to develop the definition of a more general concept, namely an $\mathcal{M}$-coseparating class for a category. Then we shall see that in a category $\mathcal{C}$ with products, a class $\mathcal{E}$ of $\mathcal{C}$-objects is an $\mathcal{M}$-coseparating class for $\mathcal{C}$ if and only if every $\mathcal{C}$-object is $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$.

PROPOSITION 3.3.1 (Herrlich and Strecker [7]). Let $\mathcal{C}$ be a category. A $\mathcal{C}$-object C is a coseparator for $\mathcal{C}$ if and only if for each $\mathcal{C}$-object A, the source $(A,\hom_{\mathcal{C}}(A,C))$ is a mono source.

PROOF: Let A be any $\mathcal{C}$-object and let p and q be any distinct $\mathcal{C}$-morphisms having the same domain and having codomain A. $(A,\hom_{\mathcal{C}}(A,C))$ is a mono source if and only if there exists $f\in\hom_{\mathcal{C}}(A,C)$ such that $f\cdot p \neq f\cdot q$, which will happen for all such $\mathcal{C}$-objects A if and only if C is a coseparator for $\mathcal{C}$.

Consequently, we know, for example, that any set with at least two points is a coseparator for Set (3.1.2). Herrlich and Strecker [7] and others have collected many examples of coseparators in categories.

Baayen [1] defined a universal object in a category $\mathcal{C}$ to be a $\mathcal{C}$-object U with the property that for every $\mathcal{C}$-object A, there exists a monomorphism $m: A \to U$. Clearly then for every $\mathcal{C}$-object A, $(A, \hom_{\mathcal{C}}(A, U))$ must be a mono source (2.3.3); hence U is a coseparator for $\mathcal{C}$. Baayen lists many examples of universal objects in categories.

DEFINITION 3.3.2. Let $\mathcal{C}$ be a category, let $\mathcal{M}$ be a class of $\mathcal{C}$-morphisms from Table 2.3.2, and let $\mathfrak{A}$ be a subcategory of $\mathcal{C}$.

(1) Let $\mathcal{E}$ be a subclass of the class of all $\mathcal{C}$-objects in $\mathcal{C}$. $\mathcal{E}$ is called an $\mathcal{M}$-<u>coseparating class for $\mathfrak{A}$</u> provided that for each $\mathfrak{A}$-object A, there exists a <u>set</u> $\mathcal{E}_A$ contained in $\mathcal{E}$ such that $(A, \bigcup_{E \in \mathcal{E}_A} \hom_{\mathcal{C}}(A,E))$ is an $\mathcal{M}$-source in $\mathcal{C}$.

(2) A $\mathcal{C}$-object E is called an $\mathcal{M}$-<u>coseparator for $\mathfrak{A}$</u> provided that $\{E\}$ is an $\mathcal{M}$-coseparating class for $\mathfrak{A}$.

From Proposition 3.3.1, it is clear that what we now call a mono-coseparator for $\mathcal{C}$ is exactly a coseparator for $\mathcal{C}$ by the usual definition. Note that for every category $\mathcal{C}$, the class $Ob(\mathcal{C})$ of all $\mathcal{C}$-objects forms an $\mathcal{M}$-coseparating class for $\mathcal{C}$, since any class $\mathcal{M}$ from Table 2.3.2 contains all the isomorphisms in $\mathcal{C}$.

Propositions 2.1.2 and 3.1.5 state the relative strengths of $\mathcal{M}$-sources for the various classes $\mathcal{M}$ of $\mathcal{C}$-morphisms from Table 2.3.2; hence they also imply the relative strengths of $\mathcal{M}$-coseparating classes for $\mathcal{C}$ for different classes $\mathcal{M}$. Therefore it is clear that every strong mono-coseparating class is an extremal mono-coseparating class, which in turn is a mono-coseparating class (2.1.2). Similarly for

complete well-powered concrete categories ($\mathcal{C}$, $\mathcal{U}$) for which $\mathcal{U}$ preserves monomorphisms and products, every extremal mono-coseparating class for $\mathcal{C}$ is a concrete embedding-coseparating class for $\mathcal{C}$, hence an extremal concrete embedding-coseparating class for $\mathcal{C}$ (3.1.5).

The following theorem is the desired result which relates $\mathcal{E}|\mathcal{M}$-embeddable objects and $\mathcal{M}$-coseparating classes. It follows immediately from Theorem 3.2.3 and the above definitions.

THEOREM 3.3.3. Let $\mathcal{C}$ be a category with products. [Let ($\mathcal{C}$, $\mathcal{U}$) be a concrete category with products such that $\mathcal{U}$ preserves monomorphisms and products.] Let $\mathcal{M}$ be a class of $\mathcal{C}$-morphisms listed in Table 2.3.2, and let $\mathcal{A}$ and $\mathcal{E}$ be subcategories of $\mathcal{C}$.

Then the following statements are equivalent:

(a) Ob($\mathcal{E}$) is an $\mathcal{M}$-coseparating class for $\mathcal{A}$.

(b) For each $\mathcal{A}$-object A, there exists a <u>set</u> $\mathcal{E}_A$ contained in Ob($\mathcal{E}$) such that the unique induced morphism $A \to \prod_{E \in \mathcal{E}_A} E^{\hom(A,E)}$ is an $\mathcal{M}$-morphism in $\mathcal{C}$.

(c) For each $\mathcal{A}$-object A, there exists a <u>set</u> $\mathcal{E}_A$ contained in Ob($\mathcal{E}$) for which there is a morphism f such that (A,f) is an $\mathcal{M}$-subobject of some product of powers of objects in $\mathcal{E}_A$.

(d) Each $\mathcal{A}$-object A is $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$.

COROLLARY 3.3.4 (Herrlich and Strecker [7]). Let $\mathcal{C}$ be a category with products and let C be a $\mathcal{C}$-object. Then the following statements are equivalent:

(a) C is a mono-coseparator (respectively, extremal mono-coseparator) for $\mathcal{C}$.

(b) For each $\mathcal{C}$-object A, the unique induced morphism $A \to C^{\hom_{\mathcal{C}}(A,C)}$ is a monomorphism (respectively, an extremal monomorphism).

(c) For each $\mathcal{C}$-object A, there is a morphism f such that (A,f) is a subobject (respectively, an extremal subobject) of some power of $\mathcal{C}$.

PROOF: A $\mathcal{C}$-object C is a mono-(respectively, an extremal mono-) coseparator for $\mathcal{C}$ if and only if {C} is a mono-(respectively, an extremal mono-) coseparating class for $\mathcal{C}$; hence the result follows from Theorem 3.3.3.

Recall that by Mrowka's result on T_1-spaces (3.2.12), $\underline{\mathrm{Top}_1}$ has no concrete embedding-coseparator in $\underline{\mathrm{Top}_1}$; however $\underline{\mathrm{Top}_1}$ has a proper concrete embedding-coseparating class in $\underline{\mathrm{Top}_1}$, namely $\mathcal{L}$, the class of all topological spaces with the finite complement topology. Note that in $\underline{\mathrm{Top}}$, the space V with three elements {a,b,c}, for which {a} is the only proper open set, is a concrete embedding coseparator for $\underline{\mathrm{Top}}$ (3.2.11) and when $\underline{\mathrm{Top}_1}$ is considered as a subcategory of $\underline{\mathrm{Top}}$, V is a concrete embedding-coseparator for $\underline{\mathrm{Top}_1}$.

DEFINITION 3.3.5. Let $\mathcal{C}$ be a category, let $\mathcal{M}$ be a class of $\mathcal{C}$-morphisms that is isomorphism-closed in $\mathcal{C}$, and let $\mathcal{E}$ be any subcategory of $\mathcal{C}$. $\mathcal{C}(\mathcal{E}|\mathcal{M})$ will be used to denote the full subcategory of $\mathcal{C}$ whose objects are precisely the $\mathcal{E}|\mathcal{M}$-embeddable $\mathcal{C}$-objects.

Clearly then Theorem 3.3.3 says that for any category $\mathcal{C}$ (respectively, concrete category $(\mathcal{C},\mathcal{U})$ such that $\mathcal{U}$ preserves monomorphisms and products) which has products, for any class $\mathcal{M}$ of $\mathcal{C}$-morphisms listed in Table 2.3.2, and for any subcategory $\mathcal{E}$ of $\mathcal{C}$, the class Ob($\mathcal{E}$) is an $\mathcal{M}$-coseparating class for $\mathcal{C}(\mathcal{E}|\mathcal{M})$.

DEFINITION 3.3.6. Let $\mathcal{C}$ be a category and let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$.

(1) Then $\mathcal{P}\mathcal{A}$ will be used to denote the full subcategory of $\mathcal{C}$ whose objects are the object part of products of $\mathcal{A}$-objects that exist in $\mathcal{C}$.

(2) Let $\mathcal{M}$ be any class of $\mathcal{C}$-morphisms that is isomorphism-closed in $\mathcal{C}$. $\mathcal{M}\mathcal{A}$ will be used to denote the full subcategory of $\mathcal{C}$ whose objects are the object parts of $\mathcal{M}$-subobjects of $\mathcal{A}$-objects.

(3) When $\mathcal{A}$ has a single object A, $\mathcal{P}$A will be used interchangeably with $\mathcal{P}\mathcal{A}$, and $\mathcal{M}$A will be used interchangeably with $\mathcal{M}\mathcal{A}$.

Then by definition of an $\mathcal{E}|\mathcal{M}$-embeddable object in a category $\mathcal{C}$ (3.2.1), the full subcategory $\mathcal{C}(\mathcal{E}|\mathcal{M})$ is precisely the full subcategory $\mathcal{M}\mathcal{P}\mathcal{E}$.

The following theorem was proved, for the class of monomorphisms in a category $\mathcal{C}$, by Herrlich and Strecker [7].

THEOREM 3.3.7. Let $\mathcal{C}$ be any category with products. Let $\mathcal{M}$ be any class of $\mathcal{C}$-morphisms that is isomorphism-closed in $\mathcal{C}$ and closed under composition and products, and let $\mathcal{A}$ and $\mathcal{B}$ be any full subcategories of $\mathcal{C}$.

(a) $\mathcal{P}\mathcal{P}\mathcal{A} = \mathcal{P}\mathcal{A}$.

(b) $\mathcal{M}\mathcal{M}\mathcal{A} = \mathcal{M}\mathcal{A}$.

(c) $\mathcal{P}\mathcal{M}\mathcal{A}$ is a full subcategory of $\mathcal{M}\mathcal{P}\mathcal{A}$.

(d) $\mathcal{M}\mathcal{P}\mathcal{A}$ is the smallest full subcategory of $\mathcal{C}$ containing $\mathcal{A}$ and closed under the formation of $\mathcal{M}$-subobjects in $\mathcal{C}$ and products in $\mathcal{C}$.

(e) $\mathcal{M}\mathcal{P}\mathcal{A}$ is a full subcategory of $\mathcal{M}\mathcal{P}\mathcal{B}$ if and only if $\mathcal{A}$

is a full subcategory of $\mathcal{MPB}$.

PROOF: (a) Clearly products can be iterated; hence $\mathcal{PP}\mathfrak{A} = \mathcal{P}\mathfrak{A}$.

(b) By hypothesis, the composition of $\mathcal{M}$-morphisms is an $\mathcal{M}$-morphism; hence $\mathcal{MM}\mathfrak{A} = \mathcal{M}\mathfrak{A}$.

(c) Let X be a $\mathcal{PM}\mathfrak{A}$-object. Then $X = \prod_{i\varepsilon I} Y_i$ for a product $(\Pi Y_i, \pi_i)$ of some family $(Y_i)_{i\varepsilon I}$ of $\mathcal{M}\mathfrak{A}$-objects, and hence for each $i\varepsilon I$, there exists an $\mathfrak{A}$-object A_i and an $\mathcal{M}$-morphism $m_i : Y_i \to A_i$. Let $(\Pi A_i, p_i)$ be the product of the family A_i, and let f be the unique induced morphism that makes the diagram

$$\begin{tikzcd} X = \Pi Y_i \arrow[r, dashed, "f"] \arrow[d, "\pi_i"'] & \Pi A_i \arrow[d, "p_i"] \\ Y_i \arrow[r, "m_i"'] & A_i \end{tikzcd}$$

commute for each $i\varepsilon I$. Then f is the product of the family $(m_i)_{i\varepsilon I}$ of $\mathcal{M}$-morphisms (1.1.4). Consequently f is an $\mathcal{M}$-morphism since $\mathcal{M}$ is closed under products. Hence X is an $\mathcal{MP}\mathfrak{A}$-object.

(d) By part (c), $\mathcal{PMP}\mathfrak{A}$ is a subcategory of $\mathcal{MPP}\mathfrak{A}$, and by (a), $\mathcal{MPP}\mathfrak{A} = \mathcal{MP}\mathfrak{A}$. Also by part (b), $\mathcal{MMP}\mathfrak{A} = \mathcal{MP}\mathfrak{A}$. So $\mathcal{MP}\mathfrak{A}$ is closed under the formation of $\mathcal{M}$-subobjects and products. Clearly any full subcategory containing $\mathfrak{A}$ and closed under the formation of $\mathcal{M}$-subobjects and products must contain $\mathcal{MP}\mathfrak{A}$.

(e) Suppose that $\mathfrak{A}$ is a full subcategory of $\mathcal{MPB}$. Then $\mathcal{MPB}$ is closed under the formation of products and $\mathcal{M}$-subobjects

(by part (d)) and hence $\mathcal{MPA}$ is a full subcategory of $\mathcal{MPB}$.

Recall that for a subcategory $\mathcal{A}$ of $\mathcal{C}$, $\mathcal{C}(\mathcal{A})$ denotes the epireflective hull of $\mathcal{A}$ in $\mathcal{C}$ (1.4.5).

COROLLARY 3.3.8. Let $\mathcal{C}$ be a complete category that is well-powered and cowell-powered, let $\mathcal{M}$ be the class of extremal monomorphisms in $\mathcal{C}$ and let $\mathcal{A}$ be a full replete subcategory of $\mathcal{C}$. Then $\mathcal{C}(\mathcal{A}|$extremal mono$)$ = $\mathcal{MPA}$ = $\mathcal{C}(\mathcal{A})$; i.e., the objects in the epireflective hull of $\mathcal{A}$ are precisely the $\mathcal{A}|\mathcal{M}$-embeddable objects in $\mathcal{C}$.

PROOF: By Theorem 1.3.5, $\mathcal{M}$ is closed under products and composition. Thus by Theorem 3.3.7 $\mathcal{MPA}$ is the smallest full subcategory of $\mathcal{C}$ containing $\mathcal{A}$ and closed under products and extremal subobjects; hence $\mathcal{MPA}$ is the smallest epireflective subcategory of $\mathcal{C}$ containing $\mathcal{A}$ (1.4.3). Thus $\mathcal{C}(\mathcal{A})$ = $\mathcal{MPA}$ = $\mathcal{C}(\mathcal{A}|$extremal mono$)$, (1.4.4 and 3.3.5).

Although the following results are simply specializations of Corollary 3.3.8 we will use them frequently throughout the remainder of this chapter.

COROLLARY 3.3.9. Let $\mathcal{A}$ be any full replete subcategory of Top. Then in Top, Top $(\mathcal{A}|$extremal mono$)$ = Top $(\mathcal{A}|$extremal concrete embedding$)$ = Top $(\mathcal{A}|$concrete embedding$)$ = Top $(\mathcal{A})$; i.e., the $\mathcal{A}$-compact spaces in Top are precisely the $\mathcal{A}$-regular spaces in Top, and these are precisely those spaces in the epireflective hull of $\mathcal{A}$ in Top.

PROOF: Top is complete, well-powered and cowell-powered. Also, the extremal monomorphisms in Top are precisely the concrete embeddings in Top ([7]).

COROLLARY 3.3.10. In Haus, let $\mathfrak{A}$ be any full, replete subcategory of Haus. Then Haus ($\mathfrak{A}$ | extremal concrete embedding) = Haus ($\mathfrak{A}$ | extremal mono) = Haus ($\mathfrak{A}$), i.e., the $\mathfrak{A}$-compact spaces in Haus are precisely those spaces in the epireflective hull of $\mathfrak{A}$ in Haus.

Furthermore Haus ($\mathfrak{A}$ | concrete embedding) = Top ($\mathfrak{A}$), i.e., the $\mathfrak{A}$-regular spaces in Haus are precisely those spaces in the epireflective hull of $\mathfrak{A}$ in Top.

PROOF: Haus is complete, well-powered and cowell-powered, and it is a full, hereditary subcategory of Top. Also the extremal concrete embeddings in Haus are precisely the extremal monomorphisms in Haus (1.5.3) and the concrete embeddings in Haus are concrete embeddings in Top (1.5.9).

§3.4 $\mathfrak{A}$-Regular and $\mathfrak{A}$-Compact Objects

CONVENTION 3.4.1.

Throughout the remainder of this chapter, all subcategories will be assumed to be both full and replete.

In the preceding section we found that for every subcategory $\mathfrak{A}$ of Top, the $\mathfrak{A}$-regular spaces in Top are precisely the spaces in the epireflective hull of $\mathfrak{A}$ in Top (3.3.9). Furthermore, we discovered that for each subcategory $\mathfrak{A}$ of Haus, the $\mathfrak{A}$-compact spaces in Haus are precisely those spaces in the epireflective hull of $\mathfrak{A}$ in Haus (3.3.10). Note that when $\mathfrak{A}$ is a subcategory of Haus, the $\mathfrak{A}$-regular spaces in Haus by our definition are all those Hausdorff spaces which are $\mathfrak{A}$-regular in Top, since concrete embeddings in Haus are concrete

embeddings in Top (1.5.9). Thus our definitions for $\mathcal{A}$-regular and $\mathcal{A}$-compact objects coincide with Herrlich's definitions in Top and Haus (§3.2). Epireflective subcategories have been studied extensively. Recall that in a complete, well-powered and cowell-powered category $\mathcal{C}$, a subcategory $\mathcal{A}$ of $\mathcal{C}$ is epireflective if and only if it is closed under the formation of products and extremal subobjects (1.4.3). Consequently, there are many examples of epireflective subcategories: Haus, Top, CompRegT_1 and Ind are epireflective in Top; CompT_2, RComp (i.e., realcompact), CompRegT_1 are epireflective in Haus; Ab is epireflective in Grp, etc.

Recall that an epireflective subcategory $\mathcal{A}$ of a category $\mathcal{C}$ has the property that for every $\mathcal{C}$-object X, there exists an $\mathcal{A}$-epireflection $(r_{\mathcal{A}}, X_{\mathcal{A}})$ in $\mathcal{C}$, where $X_{\mathcal{A}}$ is an $\mathcal{A}$-object and $r_{\mathcal{A}}: X \to X_{\mathcal{A}}$ is an $\mathcal{A}$-extendable epimorphism (1.4.2).

One of the motivating examples in the study of epireflective subcategories was the construction of the Stone-Čech compactification for completely regular T_1-spaces. Let X be a Hausdorff space and let $\mathfrak{Z} = C(X,[0,1])$. Then the product $([0,1]^{\mathfrak{Z}}, \pi_i)$ exists in Haus, and in CompT_2 (by the Tychonoff Theorem). Let the unique induced morphism be $\langle f\rangle : X \to [0,1]^{\mathfrak{Z}}$. Let $\langle f\rangle = m\cdot\beta$ be the unique (epi, extremal mono) factorization (1.3.5), and let βX be the codomain of β. Since Haus has the unique (epi, extremal mono) factorization property and since every compact space is homeomorphic to a closed subspace of the product of unit intervals (3.2.6), $(\beta,\beta X)$ can be shown to be a CompT_2-epireflection of X. Also if X is completely regular, β is a topological embedding (3.2.6).

Clearly for <u>CompT_2</u>-regular spaces, which are the completely regular spaces in <u>Haus</u>, there exist topological embeddings into compact T_2-spaces, which are 'universal' in the sense of being <u>CompT_2</u>-extendable. Herrlich and Van der Slot [10] proved a very useful result relating epireflective subcategories $\mathcal{A}$ in <u>Haus</u> and the existence of $\mathcal{A}$-epireflections $(r_{\mathcal{A}}, X_{\mathcal{A}})$ for $\mathcal{A}$-regular spaces, for which $r_{\mathcal{A}}$ is a topological embedding. The following theorem is a generalization of their result to include concrete categories other than <u>Haus</u>. This generalization and the ones following it came about because of the development of the definition for concrete embeddings.

THEOREM 3.4.2. Let $(\mathcal{C}, \mathcal{U})$ be a concrete category, which is complete, well-powered and for which $\mathcal{U}$ preserves monomorphisms. If $\mathcal{A}$ is a subcategory of $\mathcal{C}$, then each of the following statements implies the statement below it.

(a) $\mathcal{A}$ is epireflective in $\mathcal{C}$.

(b) Each $\mathcal{A}$-regular object has a concrete embedding-extension (w,W) in $\mathcal{A}$ for which w is $\mathcal{A}$-extendable.

(c) $\mathcal{A}$ is closed under the formation of products and extremal concrete embedded subobjects.

(d) $\mathcal{A}$ is closed under the formation of products and extremal subobjects.

Furthermore if $\mathcal{C}$ is cowell-powered, then the statements above, (a) through (d), are equivalent.

PROOF: <u>(c) $\Rightarrow$ (d)</u>: The extremal monomorphisms in $\mathcal{C}$ are precisely the extremal concrete embeddings in $\mathcal{C}$ (1.5.3).

<u>(a) $\Rightarrow$ (b)</u>: Let $\mathfrak{A}$ be epireflective in $\mathfrak{C}$. Suppose X is an $\mathfrak{A}$-regular object. Then there exists a product $(\Pi A_i, \pi_i)$ of $\mathfrak{A}$-objects and a concrete embedding $f: X \to \Pi A_i$. Let $(r_{\mathfrak{A}}, X_{\mathfrak{A}})$ be the $\mathfrak{A}$-epireflection of X; then $X_{\mathfrak{A}}$ is an A-object and $r_{\mathfrak{A}}: X \to X_{\mathfrak{A}}$ is an $\mathfrak{A}$-extendable epimorphism. Note that $\pi_i \cdot f$ is a morphism from X to A_i for each $i \varepsilon I$. Thus for each $i \varepsilon I$, there exists $g_i: X \to A_i$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "r_{\mathfrak{A}}"] \arrow[d, "f"'] & X_{\mathfrak{A}} \arrow[d, dashed, "g_i"] \\
\Pi A_i \arrow[r, "\pi_i"'] & A_i
\end{tikzcd}$$

commutes. Let $\langle g_i \rangle : X_{\mathfrak{A}} \to \Pi A_i$ be the unique induced morphism such that $\pi_i \cdot \langle g_i \rangle = g_i$. Then $\pi_i \cdot f = g_i \cdot r_{\mathfrak{A}} = \pi_i \cdot \langle g_i \rangle \cdot r_{\mathfrak{A}}$ for all $i \varepsilon I$; since products are mono sources, $f = \langle g_i \rangle \cdot r_{\mathfrak{A}}$. Thus $r_{\mathfrak{A}}$ is a concrete embedding, since f is (2.3.5, 2.3.3).

<u>(b) $\Rightarrow$ (c)</u>: To show that $\mathfrak{A}$ is closed under products, let $(A_i)_{i \varepsilon I}$ be any set-indexed family of $\mathfrak{A}$-objects. $\mathfrak{C}$ is complete, so the product $(\Pi A_i, \pi_i)$ is in $\mathfrak{C}$. Then ΠA_i is an $\mathfrak{A}$-regular object. Thus by (b), there exists a concrete embedding-extension (w,W), for which W is an $\mathfrak{A}$-object and $w: \Pi A_i \to W$ is an $\mathfrak{A}$-extendable epimorphism. Thus for every $i \varepsilon I$, there exists a morphism $\pi_i^*: W \to A_i$ such that the diagram

$$\begin{tikzcd}
\Pi A_i \arrow[r, "w"] \arrow[d, "\pi_i"'] & W \arrow[dl, dashed, "\pi_i^*"] \\
A_i &
\end{tikzcd}$$

commutes. Let $<\pi_i^*>:W \to \Pi A_i$ be the unique induced morphism such that $\pi_j \cdot <\pi_i^*> = \pi_j^*$ for each $j \varepsilon I$. Then $\pi_j \cdot 1_{\Pi A_i} = \pi_j^* \cdot w = \pi_j \cdot <\pi_i^*> \cdot w$ for each $j \varepsilon I$. Consequently, since products are mono sources, $1_{\Pi A_i} = <\pi_i^*> \cdot w$. Thus w is a section, as well as an epimorphism, hence an isomorphism. By hypothesis, $\mathfrak{A}$ is replete, and thus ΠA_i is an $\mathfrak{A}$-object. Also $\mathfrak{A}$ is full; so that π_i is an $\mathfrak{A}$-morphism for each $i \varepsilon I$.

To show that $\mathfrak{A}$ is closed under the formation of extremal concrete embedded subobjects, let (X,m) be an extremal concrete embedded subobject of an $\mathfrak{A}$-object A. Then X is $\mathfrak{A}$-compact, hence $\mathfrak{A}$-regular. By (b), there exists (w,W) where W is an $\mathfrak{A}$-object and $w:X \to W$ is an $\mathfrak{A}$-extendable epimorphism. Thus there exists $m^*:W \to A$ such that $m^* \cdot w = m$ (1.4.2). But m is an extremal concrete embedding, hence w is (2.3.5, 2.3.3); and consequently, w is an extremal monomorphism, as well as an epimorphism. Hence w is an isomorphism. Thus, since $\mathfrak{A}$ is replete, X is an $\mathfrak{A}$-object, and because $\mathfrak{A}$ is full, m is an $\mathfrak{A}$-morphism.

Furthermore, if $\mathfrak{C}$ is cowell-powered, the Characterization Theorem for Epireflective Subcategories (1.4.3) can be applied so that (d) $\Rightarrow$ (a).

Herrlich [5] has collected many examples in Haus which illustrate the conclusions of this theorem. We will examine an algebraic example.

EXAMPLE 3.4.3. Let AbMon be the category of Abelian monoids and monoid homomorphisms. Then Ab, the category of Abelian groups and group homomorphisms, is an epireflective subcategory of AbMon.

Let M be any Abelian monoid. We will now construct an Ab-epireflection for M, using a method outlined by Lang [12], for the construction of the Grothendieck group K(M). Let $F_{ab}(\mathcal{U}M)$ be the free Abelian group generated by $\mathcal{U}(M)$, the underlying set for M. For each $x \varepsilon M$, let [x] be the generator of $F_{ab}(\mathcal{U}M)$ corresponding to x. Clearly we have an injective function $f:M \to F_{ab}(\mathcal{U}M)$ defined by $f(x) = [x]$ for each $x \varepsilon M$. Let $B = \{[x+y] - [x] - [y] : x,y \varepsilon M\}$ and let $\langle B \rangle$ be the subgroup of $F_{ab}(\mathcal{U}M)$ generated by B. Let $\phi : F_{ab}(\mathcal{U}M) \to F_{ab}(\mathcal{U}M)/\langle B \rangle$ be the canonical homomorphism. Then $\phi \cdot f : M \to K(M)$ is a monoid homomorphism. Clearly it is an epimorphism. From the 'universal' property for free Abelian groups, it follows that $\phi \cdot f$ is Ab-extendable. Thus $(\phi \cdot f, K(M))$ is the desired Ab-epireflection for M. Let us suppose M is Ab-regular (i.e., cancellative (3.2.2)). Then there exists a set-indexed family $(A_i)_{i \varepsilon I}$ of Abelian groups, whose product is $(\Pi A_i, \pi_i)$, and there is a concrete embedding $g:M \to \Pi A_i$. But ΠA_i is an Abelian group and $\phi \cdot f$ is Ab-extendable. Thus there exists a monoid homomorphism $g^* : K(M) \to \Pi A_i$ such that $g^* \cdot \phi \cdot f = g$. Consequently, $\phi \cdot f$ is a concrete embedding since g is (2.3.5).

Recall that $\mathcal{C}(\mathcal{A} |$ concrete embedding) is the full subcategory of $\mathcal{C}$ whose objects are the $\mathcal{A}$-regular objects in $\mathcal{C}$.

THEOREM 3.4.4. Let $(\mathcal{C}, \mathcal{U})$ be a concrete category that is complete, well-powered and for which $\mathcal{U}$ preserves monomorphisms. Let $\mathcal{A}$ and $\mathcal{B}$ be epireflective subcategories of $\mathcal{C}$ such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}(\mathcal{A} |$ concrete embedding) and let X be a $\mathcal{C}$-object. If $(r_{\mathcal{A}}, X_{\mathcal{A}})$ is the $\mathcal{A}$-epireflection of X and $(r_{\mathcal{B}}, X_{\mathcal{B}})$ is the $\mathcal{B}$-epireflection of X, then there exists a concrete embedding $e:X_{\mathcal{B}} \to X_{\mathcal{A}}$ which is an $\mathcal{A}$-extendable epimorphism such that the diagram

$$\begin{tikzcd} X \arrow[rr, "r_{\mathcal{B}}"] \arrow[dr, "r_{\mathcal{A}}"'] & & X_{\mathcal{B}} \arrow[dl, dashed, "e"] \\ & X_{\mathcal{A}} & \end{tikzcd}$$

commutes.

PROOF: By hypothesis, $X_{\mathcal{A}}$ is a $\mathcal{B}$-object, since it is an $\mathcal{A}$-object. Thus, since $r_{\mathcal{B}}$ is $\mathcal{B}$-extendable (1.4.2), there exists a morphism $e:X_{\mathcal{B}} \to X_{\mathcal{A}}$ such that $r_{\mathcal{A}} = e \cdot r_{\mathcal{B}}$. But by hypothesis, $X_{\mathcal{B}}$ is $\mathcal{A}$-regular. Hence by Theorem 3.4.2, there exists a concrete embedding-extension $(\gamma, Y_{\mathcal{A}})$ in $\mathcal{A}$ for which $\gamma:X_{\mathcal{B}} \to Y_{\mathcal{A}}$ is $\mathcal{A}$-extendable. But $r_{\mathcal{A}}$ is $\mathcal{A}$-extendable and $\gamma \cdot r_{\mathcal{B}} :X \to Y_{\mathcal{A}}$; hence there exists a morphism $\alpha:X_{\mathcal{A}} \to Y_{\mathcal{A}}$ such that the exterior portion of the diagram

$$\begin{tikzcd} X \arrow[r, "r_{\mathcal{B}}"] \arrow[dr, "r_{\mathcal{A}}"'] & X_{\mathcal{B}} \arrow[r, "\gamma"] \arrow[d, "e"] & Y_{\mathcal{A}} \\ & X_{\mathcal{A}} \arrow[ur, dashed, "\alpha"'] & \end{tikzcd}$$

commutes; i.e., $\gamma \cdot r_{\mathcal{B}} = \alpha \cdot r_{\mathcal{A}}$. Thus $\gamma \cdot r_{\mathcal{B}} = \alpha \cdot e \cdot r_{\mathcal{B}}$ since $r_{\mathcal{A}} = e \cdot r_{\mathcal{B}}$. Now $\gamma = \alpha \cdot e$ since $r_{\mathcal{B}}$ is a $\mathcal{B}$-epireflection, hence an epimorphism. Thus e is a concrete embedding because γ is a concrete embedding (2.3.5). Note that e is an epimorphism because $r_{\mathcal{A}}$ is one (Duals of (2.3.5 and 2.3.3)).

To show that e is $\mathcal{A}$-extendable, suppose that there exists a morphism $f:X_{\mathcal{B}} \to A$ for some $\mathcal{A}$-object A. Since γ is $\mathcal{A}$-extendable, there exists a morphism $g^*:Y_{\mathcal{A}} \to A$ such that $g^* \cdot \gamma = g$; however, $\gamma = \alpha \cdot e$, so that $g^* \cdot \alpha \cdot e = g$. Consequently, we can conclude that e is $\mathcal{A}$-extendable (1.4.2).

Once again, the preceding theorem was proved for the category Haus by Herrlich [6], who has exhibited several examples. For instance, the category $\underline{\text{CompT}_2}$ is a full subcategory of RComp, which is a full subcategory of $\underline{\text{CompRegT}_1}$, whose objects are the $\underline{\text{CompT}_2}$-regular spaces in Haus. For a given Hausdorff space X, there exists a $\underline{\text{CompT}_2}$-epireflection $(\beta_C, \beta_C X)$ and a RComp-epireflection $(\beta_R, \beta_R X)$. Then by the conclusion of Theorem 3.4.4, $\beta_R X$ can be densely embedded in $\beta_C X$. And if X is completely regular, X can be embedded in $\beta_R X$ (3.4.2).

Now let us look at our algebraic example. Ab and CAbMon, the full subcategory of AbMon whose objects are the cancellative Abelian monoids, are epireflective subcategories of AbMon. In fact the objects of CAbMon are precisely the Ab-regular objects in AbMon (3.2.2). And Ab is a full subcategory of CAbMon. Then for any Abelian monoid M, there exists an Ab-epireflection $(\beta_A, \beta_A M)$ and an CAbMon-epireflection $(\beta_C, \beta_C M)$ of M. From Theorem 3.4.4, there exists an embedding epimorphism $e: \beta_C M \to \beta_A M$. Note that $\beta_A M$ is K(M), the Grothendieck group (3.4.3).

The following theorem was proved by Herrlich [5] for the category Haus.

THEOREM 3.4.5. Let $(\mathcal{C}, \mathcal{U})$ be a concrete category that is complete, well-powered and cowell-powered, for which $\mathcal{U}$ preserves monomorphisms. If $\mathcal{A}$ is a subcategory of $\mathcal{C}$, then the following statements are equivalent:

(a) X is an $\mathcal{A}$-regular object in $\mathcal{C}$.

(b) There exists an $\mathcal{A}$-compact object Y and a concrete embedding morphism $f: X \to Y$.

(c) There exists a concrete embedding-extension (w, W) for X where W is $\mathfrak{A}$-compact.

(d) Each $\mathfrak{A}$-extendable morphism $f: X \to Y$ is a concrete embedding.

PROOF: We will show that (b) $\Rightarrow$ (c) $\Rightarrow$ (a) $\Rightarrow$ (d) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c): Suppose Y is an $\mathfrak{A}$-compact object for which there exists a concrete embedding $f: X \to Y$. By definition of $\mathfrak{A}$-compact, there exists a product $(\Pi A_i, \pi_i)$ of $\mathfrak{A}$-objects and an extremal concrete embedding $m: Y \to \Pi A_i$. Now there exists an (epi, extremal concrete embedding) factorization of f, $f = t \cdot w$ (1.3.5, 1.5.3)

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "w"'] & & Y \arrow[r, "m"] & \Pi A_i \\
& W \arrow[ur, "t"'] & &
\end{tikzcd}$$

Let W denote the codomain of w. Then W is $\mathfrak{A}$-compact, since the composition, $m \cdot t$, of extremal monomorphisms is an extremal monomorphism in a complete, well-powered category (1.3.5) and $m \cdot t$ is a concrete embedding, since m and t are (1.5.2). Also w is a concrete embedding, because f is. Thus since w is an epimorphism and a concrete embedding, (w, W) is the required extension.

(c) $\Rightarrow$ (a): Suppose (w, W) is an $\mathfrak{A}$-compact concrete embedding-extension of X. Then since W is $\mathfrak{A}$-compact, there exists a product $(\Pi A_i, \pi_i)$ and an extremal concrete embedding $m: W \to \Pi A_i$. Also the epimorphism $w: X \to W$ is a concrete embedding since (w, W) is a concrete embedding extension. Hence $m \cdot w: X \to \Pi A_i$ is a concrete embedding (1.5.2), so X is $\mathfrak{A}$-regular.

(a) $\Rightarrow$ (d): Suppose X is $\mathcal{A}$-regular and let $f:X \to Y$ be any $\mathcal{A}$-extendable morphism. Then there exists a product $(\Pi A_i, \pi_i)$ of $\mathcal{A}$-objects and a concrete embedding $m:X \to \Pi A_i$. Thus for each $i \varepsilon I$, there exists a morphism $\pi_i \cdot m:X \to A_i$; and since f is $\mathcal{A}$-extendable, there exists a morphism $h_i:Y \to A_i$ such that the diagram

$$\begin{tikzcd}
 & \Pi A_i \arrow[dr, "\pi_i"] & \\
X \arrow[ur, "m"] \arrow[dr, "f"'] & & A_i \\
 & Y \arrow[u, dashed, "\langle h_i \rangle"'] \arrow[ur, dashed, "h_i"'] &
\end{tikzcd}$$

commutes for each $i \varepsilon I$ (1.4.2). Let the unique induced morphism be $\langle h_i \rangle:Y \to \Pi A_i$, such that $\pi_i \cdot \langle h_i \rangle = h_i$ for each $i \varepsilon I$. But $\pi_i \cdot m = h_i \cdot f$; hence $\pi_i \cdot m = \pi_i \cdot \langle h_i \rangle \cdot f$ for each $i \varepsilon I$. Thus $m = \langle h_i \rangle \cdot f$, since products are mono sources. Thus f is a concrete embedding since m is.

(d) $\Rightarrow$ (b): Suppose each $\mathcal{A}$-extendable morphism $f:X \to Y$ is a concrete embedding. In Corollary 3.3.8, we have seen that an epireflective hull $\mathcal{C}(\mathcal{A})$ for $\mathcal{A}$ exists in $\mathcal{C}$ (in fact, it is $\mathcal{C}(\mathcal{A} \mid$ extremal concrete embedding)). Thus there exists $(r_{\mathcal{C}}, X_{\mathcal{C}})$, an $\mathcal{C}(\mathcal{A})$-epireflection (1.4.2). Since $r_{\mathcal{C}}:X \to X_{\mathcal{C}}$ is $\mathcal{C}(\mathcal{A})$-extendable, it is $\mathcal{A}$-extendable, and thus it is a concrete embedding (by hypothesis). Also $X_{\mathcal{C}}$ is an $\mathcal{C}(\mathcal{A})$-object, hence $\mathcal{A}$-compact, so that (b) holds.

The following theorem has been proved by Herrlich [5] and others for the category Haus. Parts (a), (b), (c), (d) and (e) stem from the Characterization Theorem for Epireflective Hulls (1.4.6),

given by Herrlich in [6]. Once again our area of discovery lies in the interplay of $\mathfrak{A}$-regular and $\mathfrak{A}$-compact objects in categories other than Haus.

THEOREM 3.4.6. Let $(\mathfrak{C}, \mathfrak{U})$ be a concrete category that is complete, well-powered and cowell-powered, for which $\mathfrak{U}$ preserves monomorphisms. If $\mathfrak{A}$ is a subcategory of $\mathfrak{C}$, then the following statements are equivalent:

(a) X is in the epireflective hull of $\mathfrak{A}$.

(b) X is $\mathfrak{A}$-compact.

(c) Each $\mathfrak{A}$-extendable epimorphism is {X}-extendable.

(d) Each $\mathfrak{A}$-extendable epimorphism $f:X \to Y$ is an isomorphism.

(e) Each $\mathfrak{A}$-extendable morphism $f:X \to Y$ is an extremal concrete embedding.

(f) X is $\mathfrak{A}$-regular and for all $\mathfrak{A}$-regular objects Y, for which there exists an $\mathfrak{A}$-extendable concrete embedding $f:X \to Y$, f must be an extremal concrete embedding.

(g) X is $\mathfrak{A}$-regular and for each $\mathfrak{A}$-regular concrete embedding-extension (w,W) of X such that w is $\mathfrak{A}$-extendable, w must be an isomorphism.

PROOF: (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e): The extremal monomorphisms in $\mathfrak{C}$ are precisely the extremal concrete embeddings in $\mathfrak{C}$ (1.5.3). Therefore the Characterization Theorem for Epireflective Hulls can be applied (1.4.6).

(f) $\Rightarrow$ (b): Suppose X is $\mathfrak{A}$-regular. Then there exists a product $(\Pi A_i, \pi_i)$ of $\mathfrak{A}$-objects and a concrete embedding $f:X \to \Pi A_i$.

Let $\mathcal{E}(\mathcal{A})$ be the epireflective hull of $\mathcal{A}$ (1.4.4) and let $(r_{\mathcal{E}}, X_{\mathcal{E}})$ be the $\mathcal{E}(\mathcal{A})$-epireflection of X. But $\mathcal{E}(\mathcal{A})$ is $\mathcal{E}(\mathcal{A}|$extremal concrete embedding) (3.3.8) and consequently $X_{\mathcal{E}}$ is $\mathcal{A}$-compact, hence $\mathcal{A}$-regular. Note that $r_{\mathcal{E}}$ is $\mathcal{E}(\mathcal{A})$-extendable, thus $\mathcal{A}$-extendable; and for each $i \varepsilon I$, $\pi_i \cdot f : X \to A_i$ is a morphism into an $\mathcal{A}$-object. For each $i \varepsilon I$, there exists a morphism $g_i : X_{\mathcal{E}} \to A_i$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "r_{\mathcal{E}}"] \arrow[d, "f"'] & X_{\mathcal{E}} \arrow[d, dashed, "g_i"] \arrow[dl, dashed, "\langle g_i \rangle"'] \\
\Pi A_i \arrow[r, "\pi_i"'] & A_i
\end{tikzcd}$$

commutes (1.4.2). Let the unique product induced morphism be $\langle g_i \rangle : X_{\mathcal{E}} \to \Pi A_i$. Thus $\pi_i \cdot f = g_i \cdot r_{\mathcal{E}} = \pi_i \cdot \langle g_i \rangle \cdot r_{\mathcal{E}}$ for each $i \varepsilon I$. And $f = \langle g_i \rangle \cdot r_{\mathcal{E}}$ since products are mono sources. Thus $r_{\mathcal{E}}$ is a concrete embedding, since f is. And consequently by (f), $r_{\mathcal{E}}$ is an extremal concrete embedding. Hence X is $\mathcal{A}$-compact (2.3.1).

<u>(b) and (d) $\Rightarrow$ (g)</u>: Since X is $\mathcal{A}$-compact, it must be $\mathcal{A}$-regular. Let (w,W) be an $\mathcal{A}$-regular concrete embedding-extension of X for which w is $\mathcal{A}$-extendable. Then by (d), w is an isomorphism.

<u>(g) $\Rightarrow$ (f)</u>: By (g), X is $\mathcal{A}$-regular. Suppose Y is an $\mathcal{A}$-regular object for which there exists an $\mathcal{A}$-extendable concrete embedding $f : X \to Y$. Let $f = h \cdot w$ be the (epi, extremal concrete embedding) factorization of f (1.3.5, 1.5.3). Then w is a concrete embedding, since f is. Let W denote the codomain of w. Then W is $\mathcal{A}$-regular (2.3.1). Let $t : X \to A$ be a morphism to some $\mathcal{A}$-object A. Then since f

is $\mathcal{A}$-extendable there exists a morphism $k: Y \to A$ so that the diagram

$$
\begin{tikzcd}
& W \arrow[dr, "h"] & \\
X \arrow[ur, "w"] \arrow[rr, "f"] \arrow[dr, "t"'] & & Y \arrow[dl, dashed, "k"] \\
& A &
\end{tikzcd}
$$

commutes. Then $k \cdot h: W \to A$ is a morphism such that $(k \cdot h) \cdot w = t$. Hence w is $\mathcal{A}$-extendable, and (w, W) is an $\mathcal{A}$-regular concrete embedding extension. Therefore by (g), w is an isomorphism, and consequently f is an extremal concrete embedding.

4. LINEARIZATIONS

In this chapter, we will introduce the concept of $\mathcal{M}$-linearizations of an endomorphism in a category and will show that for each endomorphism in a category $\mathcal{C}$ with countable products, there exists an $\mathcal{M}$-linearization of that endomorphism for several classes $\mathcal{M}$ of $\mathcal{C}$-morphisms. We shall show further that by weakening the product condition on $\mathcal{C}$, we can still find $\mathcal{M}$-linearizations for certain endomorphisms in the category. Also, we will generalize de Groot's result [2] on the existence of 'universal linearizations' for completely regular spaces X of a weight $\leq$ k for some infinite cardinal number k and monoids M of at most k endomorphisms on X. This will become a category-theoretic result which will extend the generalizations of Baayen [1] on this same subject.

§4.1 Coordinate Immutors and Permutors

In this section, we shall develop some of the mathematical machinery for later sections. We shall define coordinate immutors and permutors on powers of objects in a category and shall see that they are endomorphisms of an interesting linear character, in that they act only on the coordinates of a power, serving to "switch or collapse" these coordinates.

DEFINITION 4.1.1. Let $\mathcal{C}$ be a category, E be a $\mathcal{C}$-object, S be any set for which the (Card S)'th power of E exists in $\mathcal{C}$, and k = Card S.

(1) Then a $\mathcal{C}$-morphism $f:E^k \to E^k$ is called a <u>coordinate immutor on</u> E^k provided that there exists a set function $\alpha:S \to S$ such

that $\pi_\phi \cdot f = \pi_{\alpha(\phi)}$ for each $\phi \varepsilon S$; i.e., f must be the unique induced morphism which makes the diagram

$$\begin{tikzcd}
E^k \arrow[r, dashed, "f"] \arrow[d, "\pi_{\alpha(\phi)}"'] & E^k \arrow[d, "\pi_\phi"] \\
E_{\alpha(\phi)} \arrow[r, equal] & E_\phi
\end{tikzcd}$$

commute for each $\phi \varepsilon S$.

(2) A coordinate immutor $f: E^k \to E^k$ is called a <u>coordinate permutor on</u> E^k provided that $\alpha: S \to S$, in the above definition, is bijective.

PROPOSITION 4.1.2. Let $\mathcal{P}$ be a set of coordinate immutors (respectively, coordinate permutors) on the object part E^k of some power in $\mathcal{C}$ of some $\mathcal{C}$-object E. If the $(k \cdot \mathrm{Card}\mathcal{P})$ 'th power of E also exists in $\mathcal{C}$, then the product of the coordinate immutors (respectively, coordinate permutors) $\prod_{p \varepsilon \mathcal{P}} p$ is a coordinate immutor (respectively, coordinate permutor), up to a natural isomorphism, on $E^{k \cdot \mathrm{Card}\mathcal{P}}$.

PROOF: Without loss of generality we may assume for each $p \varepsilon \mathcal{P}$, there exists a function $\alpha_p : k \to k$ such that $\pi_\lambda \cdot p = \pi_{\alpha_p(\lambda)}$ for each $\lambda \varepsilon k$. By hypothesis (and 1.1.3), the product $(\prod_{p \varepsilon \mathcal{P}} (E^k)_p, \overline{\pi}_p)$ is in $\mathcal{C}$.

Let $\delta: \prod_{p \varepsilon \mathcal{P}} (E^k)_p \to \prod_{\substack{\lambda \varepsilon k \\ p \varepsilon \mathcal{P}}} E_{\lambda, p}$ be the natural isomorphism, and let

$\prod_{p\in\mathcal{P}} p$ be the unique induced morphism that makes the diagram

$$\begin{tikzcd}
\prod_{p\in\mathcal{P}}(E^k)_p \arrow[r, dashed, "\prod_{p\in\mathcal{P}} p"] \arrow[d, "\bar{\pi}_p"'] & \prod_{p\in\mathcal{P}}(E^k)_p \arrow[r, shift left, "\delta"] \arrow[d, "\bar{\pi}_p"] & \prod_{\substack{\lambda\in k\\ p\in\mathcal{P}}} E_{\lambda,p} \arrow[l, shift left, "\delta^{-1}"] \arrow[ddl, "\pi_{\lambda,p}"] \\
(E^k)_p \arrow[r, "p"] \arrow[d, "(\pi_{\alpha_p(\lambda)})_p"'] & (E^k)_p \arrow[d, "(\pi_\lambda)_p"] & \\
E_{\alpha_p(\lambda),p} \arrow[r, equal] & E_{\lambda,p} &
\end{tikzcd}$$

commute for each $p\in\mathcal{P}$. [The lower portion of the diagram commutes for each $\lambda\in k$ from the definition of coordinate immutor (respectively, commutor).] $\prod_{p\in\mathcal{P}} p$ is the product of the family $(p)_{p\in\mathcal{P}}$ (1.1.4).

Define $\alpha : k\times\mathcal{P}\to k\times\mathcal{P}$ by $\alpha(\lambda,p) = (\alpha_p(\lambda),p)$ for each $\lambda\in k$, $p\in\mathcal{P}$. Then

$$\pi_{\lambda,p}\cdot\delta\cdot\left(\prod_{p\in\mathcal{P}} p\right)\cdot\delta^{-1} = (\pi_\lambda)_p\cdot\bar{\pi}_p\cdot\left(\prod_{p\in\mathcal{P}} p\right)\cdot\delta^{-1} = (\pi_\lambda)_p\cdot p\cdot\bar{\pi}_p\cdot\delta^{-1}$$

$= (\pi_{\alpha_p(\lambda)})_p\cdot\bar{\pi}_p\cdot\delta^{-1} = \pi_{\alpha((\lambda,p))}$ for each $(\lambda,p)\in k\times\mathcal{P}$; consequently $\delta\cdot\prod_{p\in\mathcal{P}} p\cdot\delta^{-1}$ is a coordinate immutor on $E^{k\cdot\mathrm{Card}\,\mathcal{P}}$ (respectively, a coordinate permutor on $E^{k\cdot\mathrm{Card}\,\mathcal{P}}$ since α is bijective, if each α_p is, for $p\in\mathcal{P}$).

PROPOSITION 4.1.3. Let $\mathcal{C}$ be a category. Let ψ be a coordinate immutor (respectively, permutor) on the object part E^k of some power of a $\mathcal{C}$-object E, and let E be the object part of the product $(\prod_{i\in I} A_i, \pi_i)$ of some family $(A_i)_{i\in I}$ of $\mathcal{C}$-objects. Then there exists an

isomorphism $\delta:E^k \to \prod_{i\in I}(A_i^k)$ such that $\delta\cdot\psi\cdot\delta^{-1}$ is equal to the product $\prod_{i\in I}\phi_i$ of a family $(\phi_i)_{i\in I}$, where each ϕ_i is a coordinate immutor (respectively, permutor) on A_i^k for each $i\varepsilon I$.

PROOF: Since ψ is a coordinate immutor (respectively, permutor) on E^k where $(E^k,(P_\lambda)_{\lambda\varepsilon k})$ is a power in $\mathfrak{C}$, then without loss of generality, we may assume that there exists a set function (respectively, a bijective set function) $\alpha:k \to k$ for which $\overline{P_\lambda}\cdot\psi = \overline{P}_{\alpha(\lambda)}$.

By the Iteration of Products Theorem (1.1.3), there exists a natural isomorphism $\delta:E^k \to \prod_{i\in I}(A_i^k)$. Consider the diagram:

$$\begin{tikzcd}
\prod_{i\in I}(A_i^k) \arrow[r, "\pi\phi_i"] \arrow[d, "\overline{\pi}_i"] \arrow[dd, bend right=60, "),i"'] & \prod_{i\in I}(A_i)^k \arrow[r, shift left, "\delta^{-1}"] \arrow[d, "\overline{\pi}_i"] \arrow[dd, bend left=60, "\pi_{\lambda,i}"] & E^k \arrow[l, shift left, "\delta"] \arrow[d, "\overline{P}_\lambda"] \arrow[dd, bend right=60, "P_{\lambda,i}"'] & E^k \arrow[l, "\psi"'] \arrow[d, "\overline{P}_{\alpha(\lambda)}"'] \arrow[dd, bend left=60, "P_{\alpha(\lambda),i}"] \\
A_i^k \arrow[r, dashed, "\phi_i"] \arrow[d, "\pi^i_{\alpha(\lambda)}"] \arrow[dr, phantom, "\mathrm{I}"] & A_i^k \arrow[d, "\pi^i_\lambda"] & E_\lambda \arrow[r, equal] \arrow[d, "P^\lambda_i"] & E_{\alpha(\lambda)} \arrow[d, "P^{\alpha(\lambda)}_i"'] \\
A_{\alpha(\lambda),i} \arrow[r, equal] & A_{\lambda,i} \arrow[r, equal] & A_{\lambda,i} \arrow[r, equal] & A_{\alpha(\lambda),i}
\end{tikzcd}$$

For each $i\varepsilon I$, let $\phi_i:A_i^k \to A_i^k$ be the morphism that makes part I of the diagram commute for each $\lambda\varepsilon k$, and let $\Pi\phi_i$ be the product of the family $(\phi_i)_{i\varepsilon I}$ (1.1.4). We need only show that $\delta\cdot\psi\cdot\delta^{-1} = \prod_{i\in I}\phi_i$. For each

$\lambda\varepsilon k$ and $i\varepsilon I$, $P_{\lambda,i}\cdot\delta^{-1}\cdot(\Pi\phi_i)\cdot\delta = \pi_{\lambda,i}\cdot(\Pi\phi_i)\cdot\delta = \pi_{\alpha(\lambda),i}\cdot\delta$

$= P_{\alpha(\lambda),i} = P_{\lambda,i}\cdot\psi$. Thus since products are mono sources (2.1.3), $\delta^{-1}\cdot(\Pi\phi_i)\cdot\delta = \psi$.

§4.2 $\mathcal{M}$-Linearizations

In this section, we will show that endomorphisms in many categories $\mathcal{C}$ can be viewed as restrictions, of some form, of coordinate immutors on powers of objects in the category, and in similar fashion, that automorphisms often can be viewed as restrictions of coordinate permutors on powers of objects. Several theorems will be proved which establish the existence of various $\mathcal{M}$-linearizations, for certain classes $\mathcal{M}$ of morphisms in a category $\mathcal{C}$ that, depending upon product properties of $\mathcal{C}$, may linearize a given individual endomorphism in $\mathcal{C}$, or simultaneously linearize a monoid of endomorphisms on a given $\mathcal{C}$-object, or even universally linearize all monoids (of a certain cardinality) of endomorphisms on an object in a given subcategory of $\mathcal{C}$. For the most part, the results of this section will generalize results of de Groot [2] and Baayen [1], but with a considerable change in emphasis.

DEFINITION 4.2.1. Let $\mathcal{C}$ be a category, $\mathcal{M}$ be a class of $\mathcal{C}$-morphisms that is isomorphism-closed in $\mathcal{C}$, $\mathcal{A}$ and $\mathcal{B}$ be subcategories of $\mathcal{C}$, X and L be $\mathcal{C}$-objects and $\phi:X \to X$ and $\psi:L \to L$ be endomorphisms in $\mathcal{C}$, and $m:X \to L$ be a $\mathcal{C}$-morphism.

(1) The triple (m,L,ψ) is called an $\mathcal{M}$-<u>lift of</u> (X,ϕ) provided that $m:X\to L$ is an $\mathcal{M}$-morphism such that the diagram

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "\phi"'] & L \arrow[d, "\psi"] \\ X \arrow[r, "m"'] & L \end{tikzcd}$$

commutes.

(2) An $\mathcal{M}$-lift (m,L,ψ) of (X,ϕ) is called an *$\mathcal{M}$-linearization of* (X,ϕ) *in* $\mathcal{P}\mathcal{A}$ provided that L is the object part of some power of a product of $\mathcal{A}$-objects and ψ is a coordinate immutor on L. If ψ is also a coordinate permutor on L, the triple (m,L,ψ) is called a *stable $\mathcal{M}$-linearization of* (X,ϕ) *in* $\mathcal{P}\mathcal{A}$.

(3) Let S be a monoid of endomorphisms on X. The triple (m,L,ψ) is called an *$\mathcal{M}$-lift of* (X,S) provided that (m,L,ψ) is an $\mathcal{M}$-lift for (X,s) for every $s \in S$.

(4) Let S be a monoid of endomorphisms on X. The triple (m,L,ψ) is called a *(stable) $\mathcal{M}$-linearization of* (X,S) in $\mathcal{P}\mathcal{A}$ provided that (m,L,ψ) is a (stable) $\mathcal{M}$-linearization of (X,s) in $\mathcal{P}\mathcal{A}$ for every $s \in S$.

Recall that each class $\mathcal{M}$, of morphisms in a category $\mathcal{C}$, listed in Table 2.3.2, is both isomorphism-closed in $\mathcal{C}$ and left-cancellative in $\mathcal{C}$ (2.3.5).

THEOREM 4.2.2. Let $\mathcal{C}$ be a category, k be a cardinal number, E be a $\mathcal{C}$-object for which the k'th power of E exists in $\mathcal{C}$, $\mathcal{M}$ be any class of $\mathcal{C}$-morphisms which is both isomorphism-closed in $\mathcal{C}$ and

left-cancellative in $\mathcal{C}$, and X be any $\mathcal{C}$-object for which there exists an $\mathcal{M}$-morphism $\xi:X \to E^k$. Let S be any monoid of endomorphisms on X (respectively, any group of automorphisms on X) for which the (k·Card S)'th power of E exists in $\mathcal{C}$.

Then for each $s\varepsilon S$, there exists a morphism $m:X \to E^{k\cdot \mathrm{Card} S}$ and an endomorphism $\gamma_s:E^{k\cdot \mathrm{Card} S} \to E^{k\cdot \mathrm{Card} S}$ such that $(m,E^{k\cdot \mathrm{Card} S},\gamma_s)$ is an $\mathcal{M}$-linearization (respectively, a stable $\mathcal{M}$-linearization) of (X,s) in $\mathcal{P}E$.

Furthermore, if $\mathcal{M}$ is contained in the class of $\mathcal{C}$-monomorphisms, then if s and s' are distinct elements of S, $\gamma_s \neq \gamma_{s'}$.

PROOF: By hypothesis the power $(\prod_{\substack{\lambda\varepsilon k\\ s\varepsilon S}} E_{\lambda,s},\pi_{\lambda,s})$ exists in $\mathcal{C}$, the power $(\prod_{\lambda\varepsilon k} E_\lambda,\pi_\lambda)$ exists in $\mathcal{C}$, and there exists an $\mathcal{M}$-morphism $\xi:X \to \Pi E_\lambda$. For each $s\varepsilon S$ and each $\lambda\varepsilon k$, there exists a morphism $\pi_\lambda\cdot\xi\cdot s:X \to E_{\lambda,\psi}$. Let m be the unique induced morphism which makes the diagram

$$\begin{tikzcd}
X \arrow[r, dashed, "m"] \arrow[d, "s"'] & \prod\limits_{\substack{\lambda\varepsilon k\\ s\varepsilon S}} E_{\lambda,s} \arrow[r, "\pi_{\lambda,s}"] & E_{\lambda,s} \arrow[d, equal] \\
X \arrow[r, "\xi"'] & \Pi E_\lambda \arrow[r, "\pi_\lambda"'] & E_\lambda
\end{tikzcd}$$

commute for each $\lambda\varepsilon k$ and $s\varepsilon S$. Thus $\pi_{\lambda,s}\cdot m = \pi_\lambda\cdot\xi\cdot s$ for each $\lambda\varepsilon k$ and $s\varepsilon S$.

We wish to show that m is an $\mathcal{M}$-morphism. For each $s\varepsilon S$, let $p_s:\Pi E_{\lambda,s} \to \Pi E_\lambda$ be the unique induced morphism such that $\pi_\lambda\cdot p_s = \pi_{\lambda,s}$ for $\lambda\varepsilon k$. However for each $\lambda\varepsilon k$ and $s\varepsilon S$, $\pi_{\lambda,s}\cdot m = \pi_\lambda\cdot\xi\cdot s$ and $\pi_\lambda\cdot p_s\cdot m = \pi_{\lambda,s}\cdot m = \pi_\lambda\cdot\xi\cdot s$. Since products are mono sources, $p_s\cdot m = \xi\cdot s$ for each $s\varepsilon S$.

By hypothesis, S is a monoid; hence $1_X\varepsilon S$. Thus $p_{1_X}\cdot m = \xi\cdot 1_X = \xi$. Moreover ξ is an $\mathcal{M}$-morphism and, by hypothesis, $\mathcal{M}$ is a left-cancellative class of $\mathcal{C}$-morphisms; hence m is an $\mathcal{M}$-morphism.

For each $y\varepsilon S$, define $\alpha_y:S \to S$ by $\alpha_y(s) = s\cdot y$ for each $s\varepsilon S$. Let t be an element of S. Let γ_t be the unique induced morphism that makes the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\prod\limits_{\substack{\lambda\varepsilon k\\ s\varepsilon S}} E_{\lambda,s} \arrow[r, dashed, "\gamma_t"] \arrow[d, "\pi_{\lambda,st}"'] & \prod\limits_{\substack{\lambda\varepsilon k\\ s\varepsilon S}} E_{\lambda,s} \arrow[d, "\pi_{\lambda,s}"] \\
E_{\lambda,s\cdot t} \arrow[r, equal] & E_{\lambda,s}
\end{tikzcd}$$

commute for each $s\varepsilon S$ and $\lambda\varepsilon k$. Clearly γ_t is a coordinate immutor.

[To show that γ_t is a coordinate permutor when S is a group of automorphisms on X, we need only show that $\alpha_t:S \to S$, as defined above, is bijective. Clearly α_t is surjective: $t^{-1}\varepsilon S$; thus for any $g\varepsilon S$,

$\alpha_t(gt^{-1}) = gt^{-1}t = g$. Suppose $\alpha_t(x) = \alpha_t(y)$ then $xt = yt$; hence $x = y$, since S is a group.]

We now will show that $(m, \prod_{\substack{\lambda\varepsilon k\\ s\varepsilon S}} E_{\lambda,s}, \gamma_t)$ is an $\mathcal{M}$-lift for (X,t); i.e., that the diagram

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "t"'] & \prod_{\substack{\lambda\varepsilon k\\ s\varepsilon S}} E_{\lambda,s} \arrow[d, "\gamma_t"] \\ X \arrow[r, "m"'] & \prod_{\substack{\lambda\varepsilon k\\ s\varepsilon S}} E_{\lambda,s} \end{tikzcd}$$

commutes.

For each $\lambda\varepsilon k$ and $s\varepsilon S$, $(\pi_{\lambda,s}\cdot\gamma_t)\cdot m = \pi_{\lambda,st}\cdot m$

$$= \pi_\lambda\cdot\xi\cdot s\cdot t$$

$$= (\pi_{\lambda,s}\cdot m)\cdot t.$$

And since products are mono sources, $\gamma_t\cdot m = m\cdot t$.

Now suppose that every $\mathcal{M}$-morphism is a monomorphism. Let $s, s'\varepsilon S$ for which $\gamma_s = \gamma_{s'}$. Then $\gamma_s\cdot m = \gamma_{s'}\cdot m$; hence $m\cdot s = m\cdot s'$; thus $s = s'$.

This theorem is a generalization of one of Baayen's results [1] which he stated only in terms of topological embeddings of completely regular T_1-spaces into powers of the real line $\mathbb{R}$. The generalization to topological embedding-linearizations in <u>Top</u> is implicit in his work.

Corollary 4.2.3 (Baayen [1]). Let F be a monoid of continuous self-maps on a completely regular T_1-space X. Then for each $f \varepsilon F$, there exists a topological embedding-linearization in some pair $(\mathbb{R}^k, \delta_f)$ where $k = (\text{Card } F) \cdot \text{weight } X \cdot \aleph_0$ and each δ_f is a continuous linear operator in $\mathbb{R}^k$. Furthermore, if f, f' are distinct elements in F, then $\delta_f \neq \delta_{f'}$.

PROOF: A well-known result of Tychonoff's states that every completely regular T_1-space X can be topologicaly embedded into the product space $[0,1]^\alpha$, where $\alpha = (\text{weight } X) \cdot \aleph_0$. Of course $[0,1]^\alpha$ is a subspace of $\mathbb{R}^\alpha$, so there exists a topological embedding $\xi : X \to \mathbb{R}^\alpha$. The class of all topological embeddings (i.e., concrete embeddings in <u>Top</u>) is left-cancellative and contained in the class of monomorphisms in <u>Top</u>. By Theorem 4.2.2, for each $f \varepsilon F$, there exists a topological embedding-linearization $(\mathbb{R}^{\alpha \cdot \text{CardF}}, \delta_f)$, such that δ_f is a coordinate immutor on $\mathbb{R}^k$, where $k = \alpha \cdot \text{CardF} = \text{weight } X \cdot \aleph_0 \cdot \text{CardF}$, and such that for some topological embedding $m : X \to \mathbb{R}^k$, the diagram

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & \mathbb{R}^k \arrow[d, "\delta_f"] \\ X \arrow[r, "m"'] & \mathbb{R}^k \end{tikzcd}$$

commutes.

There are, of course, many additional corollaries to Theorem 4.2.2. However, we will delay listing them until we have

proved a stronger version of the theorem. The next theorem allows us to simultaneously linearize a monoid of endomorphisms on an object in a category, provided that certain products exist in the category.

THEOREM 4.2.4. Let $\mathcal{C}$, $\mathcal{M}$, E, X, k and ξ be as in the hypothesis of Theorem 4.2.2. Let S be any monoid of endomorphisms (respectively, group of automorphisms) on X for which all subpowers of the $(k\cdot\text{Card } S\cdot\text{Card } S)$'th power of E exist in $\mathcal{C}$. Then there exists an $\mathcal{M}$-morphism $m:X \to E^{k\cdot\text{Card}S\cdot\text{Card}S}$ and an endomorphism ϕ on $E^{k\cdot\text{Card}S\cdot\text{Card}S}$ such that $(m,E^{k\cdot\text{Card}S\cdot\text{Card}S},\phi)$ is an $\mathcal{M}$-linearization (respectively, a stable $\mathcal{M}$-linearization) of (X,S) in $\mathcal{P}E$.

PROOF: By Theorem 4.2.2 there exists an $\mathcal{M}$-morphism $m:X \to E^{k\cdot\text{Card}S}$ such that $(m,E^{k\cdot\text{Card}S},\gamma_s)$ is an $\mathcal{M}$-linearization (respectively, a stable $\mathcal{M}$-linearization) of (X,s) in $\mathcal{P}E$. Thus the diagram

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "s"'] & E^{k\cdot\text{Card } S} \arrow[d, "\gamma_s"] \\ X \arrow[r, "m"'] & E^{k\cdot\text{Card } S} \end{tikzcd}$$

commutes. By hypothesis and the Iteration of Products Theorem (1.1.3), the product $(\Pi(E^{k\cdot\text{Card}S})_s,\pi_s)$ is in $\mathcal{C}$ and there exists a natural isomorphism $\delta\colon \prod_{s\in S} E^{k\cdot\text{Card}S} \to E^{k\cdot\text{Card}S\cdot\text{Card}S}$. Let $\prod_{s\in S}\gamma_s$ denote the product of coordinate immutors (respectively, permutors) $(\gamma_s)_{s\in S}$.

By Proposition 4.1.2, there exists $\phi = \delta\cdot\Pi\gamma_s\cdot\delta^{-1}$ which is a coordinate

immutor on $E^{k\cdot \mathrm{Card}S\cdot \mathrm{Card}S}$ (respectively a coordinate permutor on $E^{k\cdot \mathrm{Card}S\cdot \mathrm{Card}S}$).

Let $\langle m\rangle : X \to \prod_{s\in S} (E^{k\cdot \mathrm{Card}S})_s$ be the unique induced morphism for which $\pi_s\cdot\langle m\rangle = m$ for each $s\varepsilon S$. We shall show that the diagram

$$\begin{tikzcd}
X \arrow[r, "\langle m\rangle"] \arrow[d, "s"'] & \prod_{s\varepsilon S} (E^{k\cdot \mathrm{Card}\ S})_s \arrow[r, "\delta"] \arrow[d, "\pi\gamma_s"] & E^{k\cdot(\mathrm{Card}\ S)^2} \arrow[d, "\phi"] \\
X \arrow[r, "\langle m\rangle"'] & \prod_{s\varepsilon S} (E^{k\cdot \mathrm{Card}\ S})_s \arrow[r, "\delta"'] & E^{k\cdot(\mathrm{Card}\ S)^2}
\end{tikzcd}$$

commutes for each $s\varepsilon S$. $\pi_s\cdot\langle m\rangle\cdot s = m\cdot s = \gamma_s\cdot m = \gamma_s\cdot\pi_s\cdot\langle m\rangle$ $= \pi_s\cdot\pi\gamma_s\cdot\langle m\rangle$ for each $s\varepsilon S$; and, since products are mono sources, $\langle m\rangle\cdot s = \pi\gamma_s\cdot\langle m\rangle$. Thus $\langle m\rangle\cdot s = (\delta^{-1}\cdot\phi\cdot\delta)\cdot\langle m\rangle$, so that $\delta\cdot\langle m\rangle\cdot s = \phi\cdot\delta\cdot\langle m\rangle$. Also since m is an $\mathcal{M}$-morphism and $\mathcal{M}$ is a class of $\mathcal{C}$-morphisms that is left-cancellative, $\langle m\rangle$ must be an $\mathcal{M}$-morphism since $\pi_s\cdot\langle m\rangle = m$. Hence $(\langle m\rangle, E^{k(\mathrm{Card}S)^2}, \phi)$ is an $\mathcal{M}$-linearization of (X,S) in $\mathcal{P}E$ (respectively, a stable $\mathcal{M}$-linearization of (X,S) on $\mathcal{P}E$).

COROLLARY 4.2.5. Let $\mathcal{C}$ and $\mathcal{M}$ be as in the hypothesis of Theorem 4.2.4, $\mathcal{E}$ be a subcategory of $\mathcal{C}$, and X be a $\mathcal{C}$-object for which there exists an $\mathcal{M}$-morphism ξ from X into the product $(\prod_{i\varepsilon I} E_i, \pi_i)$ of a set-indexed family of $\mathcal{E}$-objects. Let S be a monoid of endomorphisms on X (respectively, a group of automorphisms on X) such that all subpowers of the $(\mathrm{Card}S\cdot \mathrm{Card}S)$'th power of $(\prod_{i\varepsilon I} E_i)$ exist in $\mathcal{C}$.

Then there exists an $\mathcal{M}$-morphism m and an endomorphism ϕ on $(\prod_{i\in I} E_i)^{(\mathrm{Card}S)^2}$ such that $(m,(\prod_{i\in I} E_i)^{(\mathrm{Card}S)^2},\phi)$ is an $\mathcal{M}$-linearization (respectively, a stable $\mathcal{M}$-linearization) of (X,S) in $\mathcal{PE}$.

PROOF: Let $E = \prod_{i\in I} E_i$. Then apply Theorem 4.2.4.

COROLLARY 4.2.6. Let k be any infinite cardinal number, $\mathcal{C}$ be a category with k-fold products, $\mathcal{M}$ be a class of morphisms in $\mathcal{C}$ that is isomorphism-closed and left-cancellative in $\mathcal{C}$, $\mathcal{E}$ be a full subcategory of $\mathcal{C}$, and X be a $\mathcal{C}$-object. Then the following statements are equivalent:

(a) X is $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$.

(b) For any monoid S of endomorphisms on X for which Card S $\leq$ k, there exists an $\mathcal{M}$-linearization of (X,S) in $\mathcal{PE}$.

(c) For any endomorphism t on X, there exists an $\mathcal{M}$-linearization of (X,t) in $\mathcal{PE}$.

(d) For any endomorphism t on X of finite order, there exists an $\mathcal{M}$-linearization of (X,t) in $\mathcal{PE}$.

(e) For any automorphism t on X, there exists a stable $\mathcal{M}$-linearization of (X,t) in $\mathcal{PE}$.

(f) For any automorphism t on X of finite order, there exists an $\mathcal{M}$-linearization of (X,t) in $\mathcal{PE}$.

(g) For any group S of automorphisms on X for which Card S $\leq$ k, there exists a stable $\mathcal{M}$-linearization of (X,S) in $\mathcal{PE}$.

PROOF: <u>(a) $\Rightarrow$ (b)</u> (respectively <u>(a) $\Rightarrow$ (g)</u>): There exists a set-indexed family $(E_i)_{i\in I}$ of $\mathcal{E}$-objects for which the product $(\prod E_i,\pi_i)$ exists in

$\mathcal{C}$ and there is an $\mathcal{M}$-morphism $\xi : X \to \Pi E_i$ (3.2.1). Let S be any monoid of endomorphisms (respectively group of automorphisms on X) for which Card S $\leq$ k. Then $(\text{Card } S)^2 \leq k$. Apply Corollary 4.2.5.

(b) $\Rightarrow$ (c) (respectively (g) $\Rightarrow$ (e)): Let S_T be the monoid (respectively, group) generated by $\{t\}$. Then Card $S_T \leq \aleph_0 \leq k$.

(a) $\Rightarrow$ (d) and (e) $\Rightarrow$ (f): Clear.

(d) $\Rightarrow$ (a) (respectively (f) $\Rightarrow$ (a)): The identity 1_X has order 1. Thus there exists a (stable) $\mathcal{M}$-linearization of $(X, 1_X)$ in $\mathcal{P}\mathcal{E}$. Thus there exists L, a power of $\mathcal{E}$-objects, and an $\mathcal{M}$-morphism $m : X \to L$ (4.2.1). Hence X is $\mathcal{E}|\mathcal{M}$-embeddable.

COROLLARY 4.2.7. Let $\mathcal{C}$ be a category with products, let $\mathcal{M}$ be a class of $\mathcal{C}$-morphisms that is isomorphism-closed in $\mathcal{C}$ and left-cancellative in $\mathcal{C}$. Let $\mathcal{E}$ be any subcategory of $\mathcal{C}$. Then X is $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$ if and only if for every monoid S of endomorphisms on X (respectively, every group S of automorphisms on X), there exists an $\mathcal{M}$-linearization of (X,S) (respectively, a stable $\mathcal{M}$-linearization of (X,S)) in $\mathcal{P}\mathcal{E}$.

PROOF: Apply the previous corollary.

There are three areas of differences between Baayen's generalizations of de Groot's work and our generalizations. The first is primarily one of emphasis: Baayen [1] was interested in the existence of mono-universal objects in a category and universal mono-lifts for morphisms in the category (i.e., "universal morphisms"), while our emphasis is on the linear character of the resulting coordinate immutors. Secondly, we have obtained results on the existence of $\mathcal{M}$-linearizations for several classes $\mathcal{M}$ of morphisms in

a category, while Baayen's results were restricted to monomorphisms in general categories and topological embeddings in <u>Top</u> (which he had to consider separately). Thirdly, Baayen's results were restricted to categories with countable products; our results require only the existence of certain products in the category. Hence, for example, we can obtain the following corollary for categories with finite products.

COROLLARY 4.2.8. Let $\mathcal{C}$ be a category with finite products, $\mathcal{M}$ be any class of $\mathcal{C}$-morphisms that is isomorphism-closed in $\mathcal{C}$ and left-cancellative in $\mathcal{C}$, $\mathcal{E}$ be a subcategory of $\mathcal{C}$, and X be a $\mathcal{C}$-object. Then the following statements are equivalent:

(a) X is $\mathcal{E}|\mathcal{M}$-embeddable in $\mathcal{C}$.

(b) For any monoid S of endomorphisms on X for which Card S is finite, there exists an $\mathcal{M}$-linearization of (X,S) in $\mathcal{PE}$.

(c) For any endomorphism t on X of finite order, there exists an $\mathcal{M}$-linearization of (X,t) in $\mathcal{PE}$.

(d) For any automorphism g on X of finite order, there exists a stable $\mathcal{M}$-linearization of (X,g) in $\mathcal{PE}$.

(e) For any group S of automorphisms on X for which Card S is finite, there exists a stable $\mathcal{M}$-linearization of (X,S) in $\mathcal{PE}$.

PROOF: <u>(a) $\Rightarrow$ (b)</u> (respectively, <u>(a) $\Rightarrow$ (e)</u>): There exists a product $(\prod_{i\in I} E_i, \pi_i)$ of $\mathcal{E}$-objects and an $\mathcal{M}$-morphism $\xi: X \to \prod_{i\in I} E_i$. $(\text{Card } S)^2$ is finite, since Card S is finite. Apply Corollary 4.2.5 to obtain the $\mathcal{M}$-linearization of (X,S) in $\mathcal{PE}$ (respectively, to obtain the stable $\mathcal{M}$-linearization of (X,S) in $\mathcal{PE}$).

(b) $\Rightarrow$ (c) (respectively, (e) $\Rightarrow$ (d)): Let S_T be the monoid of endomorphisms on X (respectively, group of automorphisms on X) generated by $\{t\}$. Since t has finite order, Card S_T is finite; hence by (b) there exists an $\mathcal{M}$-linearization (respectively, a stable $\mathcal{M}$-linearization) (m,L,ϕ) of (X,S_T) in $\mathcal{P}\mathcal{E}$. Hence (m,L,ϕ) is the desired linearization for (X,t).

(c) $\Rightarrow$ (a) (respectively, (d) $\Rightarrow$ (a)): The identity morphism on X has order 1. Therefore by (c) (respectively, (d)), there exists an $\mathcal{M}$-linearization (m,L,ϕ) of $(X,1_X)$ in $\mathcal{P}\mathcal{E}$, where L is a power of a product of $\mathcal{E}$-objects.

A much stronger result can be obtained for a category $\mathcal{C}$ with countable products: every endomorphism ϕ on a $\mathcal{C}$-object X has a section-linearization in $\mathcal{P}$X, and hence an $\mathcal{M}$-linearization in $\mathcal{P}$X for every class $\mathcal{M}$ of $\mathcal{C}$-morphism from Table 2.3.2 (2.3.7).

COROLLARY 4.2.9. In a category $\mathcal{C}$ with countable products, every $\mathcal{C}$-object X and every endomorphism t on X (respectively, every automorphism t on X) has a section-linearization of (X,t) in $\mathcal{P}$X.

PROOF: Let S_t be the monoid (respectively, group) generated by $\{t\}$. Card $S_t \leq \aleph_0$. The identity morphism $1_X : X \to X$ is an isomorphism, hence a section, and the class of all sections in $\mathcal{C}$ is left-cancellable and isomorphism-closed in $\mathcal{C}$. Thus there exists a section-linearization of (X,t) in $\mathcal{P}$X (respectively, a stable section-linearization of (X,t) in $\mathcal{P}$X) (4.2.6).

It is interesting to note that, in a category $\mathcal{C}$, with countable products and a weakly terminal class $\mathcal{E}$ of $\mathcal{C}$-objects

(i.e., for each $\mathcal{C}$-object X, $\hom_{\mathcal{C}}(X,E) \neq \phi$ for some $E \in \mathcal{E}$), every pair (X,ϕ), where X is a $\mathcal{C}$-object and ϕ is an endomorphism on X, has a weak-linearization in $\mathcal{P}\mathcal{E}$. If a category $\mathcal{C}$ has an epireflective subcategory $\mathcal{A}$, then $\mathrm{Ob}(\mathcal{A})$ is a weakly terminal class for $\mathcal{C}$ (1.4.2). Consequently if $(\mathcal{C},\mathcal{U})$ is a concrete category that is complete and well-powered, having a full replete epireflective subcategory $\mathcal{A}$, then $\mathcal{P}\mathcal{A} = \mathcal{A}$ (1.4.3) and for every $\mathcal{C}$-object X and every endomorphism ϕ on X, there exists a weak-linearization of (X,ϕ) in $\mathcal{A}$; and furthermore, if X is an $\mathcal{A}$-regular space, there exists a concrete embedding-linearization of (X,ϕ) in $\mathcal{A}$ (3.2.1, 4.2.6).

Let us next consider a product $B\times C$ of distinct sets B and C and a Set-monomorphism $m:X \to B\times C$ and an automorphism $f:X \to X$. We know that there exists a coordinate permutor $\psi:(B\times C)^{\alpha} \to (B\times C)^{\alpha}$, where α is the order of the automorphism f, such that $(m,(B\times C)^{\alpha},\psi)$ is a stable mono-linearization of (X,f) in Set (4.2.6). What we will examine now is the workings of ψ on $(B\times C)^{\alpha}$. From the next theorem we will find that $\psi = \psi_B\times\psi_C$, the product of coordinate permutors ψ_B on B^{α} and ψ_C on C^{α}.

THEOREM 4.2.10. Let $\mathcal{C}$ be a category, let $\mathcal{M}$ be a class of $\mathcal{C}$-morphisms that is isomorphism-closed in $\mathcal{C}$ and left-cancellative in $\mathcal{C}$, let $(E_i)_{i\in I}$ be a set-indexed family of $\mathcal{C}$-objects whose product $(\Pi E_i,\pi_i)$ exists in $\mathcal{C}$, and let X be any $\mathcal{C}$-object for which there exists an $\mathcal{M}$-morphism $\xi:X \to \Pi E_i$. If S is any monoid of endomorphisms on X (respectively, any group of automorphisms on X) such that all subpowers of the (Card S·Card S)'th power of ΠE_i exist in $\mathcal{C}$, then there exists a triple (m,L,ψ) which is an $\mathcal{M}$-lift for (X,S) such that

$L = \prod_{i\varepsilon I} E_i^{(\mathrm{Card}S)^2}$ and ψ is the product of a family $(\phi_i)_{i\varepsilon I}$ of morphisms, where for each $i\varepsilon I$, ϕ_i is a coordinate immutor (respectively, permutor) on $E_i^{(\mathrm{Card}S)^2}$.

PROOF: By Theorem 4.2.4, there exists an $\mathcal{M}$-linearization (respectively, stable $\mathcal{M}$-linearization) $(m,(\Pi E_i)^{(\mathrm{Card}S)^2},\psi)$ for $(\Pi E_i,S)$. The remainder of the proof follows directly from Proposition 4.1.3.

§4.3 Universal $\mathcal{M}$-Linearizations

In the previous section, we found that linearizations always exist for endomorphisms in categories with countable products, and that for some monoids of endomorphisms on an object in the category, as well as for some groups of automorphisms on an object in the category, we could obtain simultaneous linearizations. In this section, we will restrict our attention to categories with infinite products in order to obtain some linearizations that are universal for all endomorphisms (respectively, all automorphisms) on any object in a given subcategory.

DEFINITION 4.3.1. Let $\mathcal{C}$ be a category, $\mathcal{M}$ be a class of $\mathcal{C}$-morphisms that is isomorphism-closed in $\mathcal{C}$, $\mathcal{O}$ be a subcategory of $\mathcal{C}$ and L be a $\mathcal{C}$-object where $\psi:L \to L$ is an endomorphism in $\mathcal{C}$.

(1) A $\mathcal{C}$-object X is called an $\mathcal{M}$-<u>universal object for</u> $\mathcal{O}$ provided that for each $\mathcal{O}$-object A there exists an $\mathcal{M}$-morphism $m:A \to X$.

(2) A pair (L,ψ) is called a <u>universal</u> $\mathcal{M}$-<u>lift for</u> <u>End</u>$(\mathcal{O})$ [respectively, <u>for</u> <u>Aut</u>$(\mathcal{O})$] provided that for each endomorphism [respectively, each automorphism] ϕ on any $\mathcal{O}$-object A, there exists an

$\mathcal{M}$-morphism $m:A \to L$ for which (m,L,ψ) is an $\mathcal{M}$-lift for (A,ϕ).

(2) A universal $\mathcal{M}$-lift (L,ψ) for End($\mathcal{A}$) [respectively, for Aut($\mathcal{A}$)] is called a universal $\mathcal{M}$-linearization for End($\mathcal{A}$) [respectively, for Aut($\mathcal{A}$)] provided that L is the object part of a power of a product of $\mathcal{C}$-objects and ψ is a coordinate immutor. If ψ is a coordinate permutor on L, then (L,ψ) is called a stable universal $\mathcal{M}$-linearization for End($\mathcal{A}$) [respectively, for Aut($\mathcal{A}$)].

(3) Let k be a cardinal number. A monoid S is called a k-$\mathcal{M}$-universal monoid for $\mathcal{A}$ provided that card $S \leq k$ and there exists a $\mathcal{C}$-object L and a monoid homomorphism $\alpha:S \to \hom_{\mathcal{C}}(L,L)$ with the following property: for each monoid T with card $T \leq k$, there exists a surjective monoid homomorphism $\gamma:S \to T$ such that for each $\mathcal{A}$-object A and for each monoid homomorphism $\beta:T \to \hom_{\mathcal{C}}(A,A)$ there is some $\mathcal{M}$-morphism $m:A \to L$, such that for each $s \in S$, the triple $(m,L,\alpha(s))$ is an $\mathcal{M}$-lift of $(A,\beta\cdot\gamma(s))$.

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "\beta\cdot\gamma(s)"'] & L \arrow[d, "\alpha(s)"] \\ A \arrow[r, "m"'] & L \end{tikzcd}$$

(4) Let k be a cardinal number. A group S is called a k-$\mathcal{M}$-universal group for $\mathcal{A}$ provided that card $S \leq k$ and there exist a $\mathcal{C}$-object L and a group homomorphism $\alpha:S \to \mathrm{Aut}_{\mathcal{C}}(L,L)$ (where

$Aut_{\mathcal{C}}(L,L)$ is the group of automorphisms on L) with the following property:

for each group T with card $T \leq k$, there exists a surjective group homomorphism $\gamma:S \to T$ such that for each $\mathcal{A}$-object A and for each group homomorphism $\beta:T \to Aut_{\mathcal{C}}(A,A)$ there is some $\mathcal{M}$-morphism $m:A \to L$, such that for each $s \in S$ the triple $(m,L,\alpha(s))$ is an $\mathcal{M}$-lift of $(A,\beta\cdot\gamma(s))$.

THEOREM 4.3.2. Let k be an infinite cardinal number, $\mathcal{C}$ be a category with k-fold products, and $\mathcal{M}$ be any class of $\mathcal{C}$-morphisms that is isomorphism-closed in $\mathcal{C}$ and left-cancellative in $\mathcal{C}$. Then if $\mathcal{A}$ is any subcategory of $\mathcal{C}$, the following statements are equivalent:

(a) There exists an $\mathcal{M}$-universal object for $\mathcal{A}$ in $\mathcal{C}$.

(b) There exists a k-$\mathcal{M}$-universal monoid for $\mathcal{A}$ in $\mathcal{C}$.

(c) There exists a k-$\mathcal{M}$-universal group for $\mathcal{A}$ in $\mathcal{C}$.

(d) There exists an universal $\mathcal{M}$-linearization for End($\mathcal{A}$) in $\mathcal{C}$.

(e) There exists a stable universal $\mathcal{M}$-linearization for Aut($\mathcal{A}$) in $\mathcal{C}$.

(f) There exists an universal $\mathcal{M}$-lift for End($\mathcal{A}$) in $\mathcal{C}$.

Moreover if $\mathcal{A}$ is closed under the formation of k-fold products in $\mathcal{C}$, then $\mathcal{C}$ in statements (a) through (f) may be replaced by $\mathcal{A}$.

PROOF: (a) $\Rightarrow$ (b): Let S be a free monoid with k generators. Card S = $k\cdot\aleph_0 = k$, since k is infinite. Let K be the $\mathcal{M}$-universal object for $\mathcal{A}$. By hypothesis, the product $(K^k, (\pi_\lambda)_{\lambda\epsilon k})$ exists in $\mathcal{C}$. For each $t \epsilon S$, define $f_t:K^k \to K^k$ to be the unique induced morphism that makes the diagram

$$\begin{tikzcd}
K^k \arrow[r, dashed, "f_t"] \arrow[d, "\pi_{st}"'] & K^k \arrow[d, "\pi_s"] \\
K_{st} \arrow[r, equal] & K_s
\end{tikzcd}$$

commute for each $s \varepsilon S$. Define $\alpha: S \to \hom_{\mathcal{C}}(K^k, K^k)$ by $\alpha(t) = f_t$ for each $t \varepsilon S$. Note that for given $p, q \varepsilon S$ and for every $s \varepsilon S$, $\pi_s \cdot f_p \cdot f_q = \pi_{sp} \cdot f_q = \pi_{spq} = \pi_s \cdot f_{pq}$, and since products are mono sources (2.1.3), $\alpha(p)\alpha(q) = f_p \cdot f_q = f_{pq} = \alpha(pq)$. Also, clearly $\alpha(1) = f_1$, where $\pi_s \cdot f_1 = \pi_{s \cdot 1} = \pi_s$ for every $s \varepsilon S$; hence $\alpha(1) = 1_{K^k}$. Thus α is a monoid homomorphism.

Let T be any monoid such that Card $T \leq k$. By the universal mapping property for free monoids, there exists a surjective monoid homorphism $\gamma: S \to T$. For some $\mathfrak{A}$-object A, let $\beta: T \to \hom_{\mathcal{C}}(A, A)$ be a monoid homomorphism. Since K is an $\mathcal{M}$-universal object for $\mathfrak{A}$, there exists an $\mathcal{M}$-morphism $m: A \to K$. Thus for each $s \varepsilon S$, there exists a morphism $m \cdot \beta(\gamma(s)): A \to K$. Let $q: A \to K^k$ be the unique induced morphism such that the diagram

$$\begin{tikzcd}
A \arrow[rr, dashed, "q"] \arrow[d, "\beta(\gamma(s))"'] & & K^k \arrow[d, "\pi_s"] \\
A \arrow[r, "m"] & K \arrow[r, equal] & K_s
\end{tikzcd}$$

commutes for each $s \varepsilon S$. Thus $\pi_s \cdot q = m \cdot \beta(\gamma(s))$ for each $s \varepsilon S$. T and S are monoids; hence there exist identities $e_S \varepsilon S$ and $e_T \varepsilon T$ and $\gamma(e_S) = e_T$. Thus $\beta(\gamma(e_S)) = \beta(e_T) = 1_A$, and $\pi_{e_S} \cdot q = m \cdot 1_A = m$. $\mathcal{M}$ is left-cancellative in $\mathcal{C}$; thus q is an $\mathcal{M}$-morphism, since m is.

Now we need only to show that the diagram

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "\beta\cdot\gamma(t)"'] & K^k \arrow[d, "\alpha(t)"] \\ A \arrow[r, "q"'] & K^k \end{tikzcd}$$

commutes for each $t \varepsilon S$. For every $s \varepsilon S$, $\pi_s \cdot \alpha(t) \cdot q = \pi_s \cdot f_t \cdot q = \pi_{st} \cdot q = m \cdot \beta(\gamma(st)) = m \cdot \beta(\gamma(s)) \cdot \beta(\gamma(t)) = \pi_s \cdot q \cdot \beta(\gamma(t))$. But products are mono sources (2.1.3), so $\alpha(t) \cdot q = q \cdot \beta(\gamma(t))$ for each $t \varepsilon S$. Hence $(q, K^k, \alpha(t))$ is an $\mathcal{M}$-lift of $(A, \beta \cdot \gamma(t))$ for each $t \varepsilon S$.

<u>(b) ⇒ (a)</u>: Let S be a k-$\mathcal{M}$-universal monoid for $\mathfrak{A}$ and let $\alpha : S \rightarrow \hom_{\mathcal{C}}(L,L)$ be the monoid homorphism with the property defined in 4.3.1. Let T be a monoid with one element $\{x\}$. For each $\mathfrak{A}$-object A, let $\beta_A : T \rightarrow \hom_{\mathcal{C}}(A,A)$ be defined by $\beta_A(x) = 1_A$. Then by 4.3.1, there must exist $m : A \rightarrow L$. Thus L is an universal object for $\mathfrak{A}$.

<u>(a) ⇒ (c)</u>: This proof is analogous to the proof for (a) ⇒ (b). Let S be the free group with k generators. Define $\alpha : S \rightarrow \mathrm{Aut}_{\mathcal{C}}(K^k, K^k)$ as in the proof for (a) ⇒ (b). We need only show that $\alpha(t) = f_t$ has an inverse for each $t \varepsilon T$. $\alpha(t^{-1}) \cdot \alpha(t) = f_{t^{-1}} \cdot f_t = f_{t^{-1} \cdot t} = \alpha(t^{-1} \cdot t) = 1_{K^k} = \alpha(t \cdot t^{-1}) = \alpha(t)\alpha(t^{-1})$, since S is a group.

<u>(c) $\Rightarrow$ (a)</u>: The proof is analogous to the proof for (b) $\Rightarrow$ (a).

<u>(a) $\Rightarrow$ (d)</u>: Let S be the free monoid with k generators, let K be the $\mathcal{M}$-universal object for $\mathfrak{A}$, and construct $\alpha : S \to \hom_{\mathcal{C}}(K^k, K^k)$ as in the proof for (a) $\Rightarrow$ (b). Clearly $\alpha(s)$ is a coordinate immutor for each $s \varepsilon S$. Let T be the free monoid with one generator 'x.' As before, there exists a surjective monoid homomorphism $\gamma : S \to T$. Thus for some $s_x \varepsilon S$, $\gamma(s_x) = x$. We shall now show that $(K^k, \alpha(s_x))$ is a universal $\mathcal{M}$-linearization for $\operatorname{End}(\mathfrak{A})$. Let A be any $\mathfrak{A}$-object and ϕ be any endomorphism on A. Since K is an $\mathcal{M}$-universal object for $\mathfrak{A}$, there is some $\mathcal{M}$-morphism $m_A : A \to L$. Define $\beta_A : T \to \hom_{\mathcal{C}}(A, A)$ by $\beta_A(x) = \phi$. Then let $q_A : A \to K^k$ be the unique induced morphism that makes the diagram

$$\begin{tikzcd}
A \arrow[rr, dashed, "q_A"] \arrow[d, "\beta \cdot \gamma(s)"'] & & K^k \arrow[d, "\pi_s"] \\
A \arrow[r, "m_A"] & K \arrow[r, equal] & K
\end{tikzcd}$$

commute for each $s \varepsilon S$. Then, as before, for every $s \varepsilon S$ the diagram

$$\begin{tikzcd}
A \arrow[r, "q_A"] \arrow[d, "\beta_A(\gamma(s))"'] & K^k \arrow[d, "\alpha(s)"] \\
A \arrow[r, "q_A"'] & K^k
\end{tikzcd}$$

commutes, but $\beta_A(\gamma(s_x)) = \beta_A(x) = \phi$. Hence $(q_A, K^k, \alpha(s_x))$ is an $\mathfrak{M}$-lift for (A,ϕ).

(d) $\Rightarrow$ (f): Clear.

(a) $\Rightarrow$ (e): The proof is analogous to the proof for (a) $\Rightarrow$ (d). We need only check that $\alpha(t)$ as defined is a coordinate permutor for each $t \in S$. Let $\rho: S \to S$ be defined by $\rho(s) = st$. Now, since S is a group, ρ is bijective. Thus $\alpha(t)$ is a coordinate permutor (4.1.1) for each $t \in S$. Hence $(K^k, \alpha(s_x))$ is a stable universal $\mathfrak{M}$-linearization for all automorphisms in $\mathfrak{C}$.

(e) $\Rightarrow$ (f): Clear.

(f) $\Rightarrow$ (a): Clear.

Finally we have de Groot's result which initiated the investigations of linearizations and universal objects in categories.

COROLLARY 4.3.3 (de Groot [2]). Let k be an infinite cardinal number, let P be the topological product $[0,1]^k$ and let F be the free monoid with k generators. Then every completely regular T_1-space X of weight $\leq$ k and every monoid S of endomorphisms on X admits a universal linearization in the pair (F,P) (i.e., in our terminology, there exists a k-topological embedding-monoid for the subcategory of Top whose objects are the completely regular T_1-spaces of weight $\leq$ k).

PROOF: By Tychonoff's result, $P = [0,1]^k$ is a universal object for all completely regular T_1-spaces of weight $\leq$ k. The class of all topological embeddings in Top is precisely the class of all concrete embeddings in Top, thus it is left-cancellative (2.3.5). Consequently Theorem 4.3.2 may be applied.

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